Problem 1. In the "Naboj" country, people are using *nabojs* as currency. There are nine types of banknotes of values 1 *naboj*, 10 *nabojs*, 100 *nabojs*, 1 000 *nabojs*, 10 000 *nabojs*, 100 000 *nabojs*, 1 000 000 *nabojs*, 10 000 000 *nabojs* and 100 000 000 *nabojs*. Joseph has one banknote of each kind. What is the arithmetic mean of the banknotes that Joseph has?

Result. Arithmetic mean of all Joseph´s banknotes is 12 345 679 *nabojs*.

Solution. The arithmetic mean is the sum of a set of numbers divided by the count of numbers in the set. We can compute the arithmetic mean of values of Joseph´s banknotes by summing all values of these banknotes (it is 111 111 111) and then dividing it by the number of banknotes (9). The arithmetic mean is $111\,111\,111/9 = 12\,345\,679$.

In order to make division easier, we can write each banknote as follows: $1 = 1$, $10 = 9 + 1$, $100 = 99 + 1$, $\ldots$, $100\,000\,000 = 99\,999\,999 + 1$. We can also write sum of all these values in a form $9 + 99 + 999 + \cdots + 99\,999\,999 + 9 \cdot 1$. When we divide this number by 9, the result will be $1 + 11 + 111 + \cdots + 11\,111\,111 + 1 = 12\,345\,679$, which is the desired arithmetic mean.

Problem 2. Roberta has just missed a bus. Since she expected it, she is wearing jogging shoes. Her school is 1 kilometer away from the bus stop, and Roberta's lesson starts in 5 minutes. What is the minimum average speed Roberta must run in order to be at school on time?

Result. Roberta's average speed must be at least $12\,\mathrm{km/h} = 10/3\,\mathrm{m/s} \doteq 3.33\,\mathrm{m/s}$.

Solution. The average speed of any motion is the ratio of the total covered distance and the time required to cover this distance. In this case, Roberta must run at least 1 kilometer in 5 minutes. Nevertheless, we need speed in more standard units, for example, kilometers per hour. Five minutes are equal to one-twelfth of an hour. We can realize that if Roberta runs 1 kilometer in 5 minutes, then in 12 such 5 minute sections, she will cover 12 kilometers. That means her average speed is at least $12\,\mathrm{km/h}$.

Problem 3. Catherine is painting a wall in her room. The wall is $2.5\,\mathrm{m}$ high and $6\,\mathrm{m}$ wide. It takes exactly 2 minutes to paint one square meter of wall. How long will Catherine paint the wall?

Result. Painting the wall will take 30 minutes = 0.5 hour.

Solution. Let us compute the total wall area that Catherine needs to paint. It is $2.5\,\mathrm{m} \cdot 6\,\mathrm{m} = 15\,\mathrm{m}^2$. Using the fact that $1\,\mathrm{m}^2$ takes two minutes to paint, we get that Catherine needs $15 \cdot 2\,\mathrm{min} = 30\,\mathrm{min}$ to pain the whole wall.

Problem 4. *Word search* is a game in which one searches words in a square grid. Each tile of this square grid contains a single letter. The words might be placed in any of 8 directions. In how many ways can the word *NABOJ* be placed in word search formed by 5×5 grid? The picture shows you some possible ways how to place *NABOJ* in grid 5×5.

J				
O				
B				
A				
N				

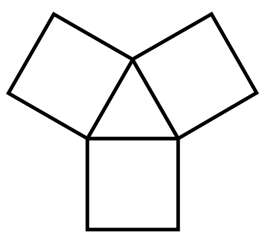
N	A	B	O	J

				N
			A	
		B		
	O			
J				

Result. There are 24 ways, how to place word *NABOJ* into the word search.

Solution. If it is possible to place the word into word search in some direction, it is also possible to place the word in the opposite direction. That means we can reduce possible directions from 8 to 4 – two diagonal, one vertical, and one horizontal. The final number of ways will be two times the ways we will find. There are two diagonals – from top right to bottom left and from top left to bottom right. Because of that, there is $2 \cdot 2 = 4$ ways how to place word *NABOJ* onto the diagonal of the grid (this includes also diagonals with opposite direction). For vertical direction, there are 5 ways, also 5 ways for the horizontal direction. Including opposite directions, the number of ways is $2 \cdot 5 + 2 \cdot 5 = 20$. The number of all of the possible ways how to place word *NABOJ* into the word search is then $4 + 20 = 24$.

Problem 5. Luke drew an equilateral triangle with a side of length 6 cm. Then he drew three squares, as illustrated in the picture. What is the circumference of Luke's figure?



Result. Circumference of Luke's figure was 54 cm.

Solution. All three squares share one common side with the inner equilateral triangle. The side length of this equilateral triangle is 6 cm, so the side length of each square must be 6 cm, too. Each of the squares contributes to the total circumference with three of its sides i.e., with length $3 \cdot 6\,\text{cm} = 18\,\text{cm}$. There are three such squares, so the circumference of the whole figure is $3 \cdot 18\,\text{cm} = 54\,\text{cm}$.

Problem 6. Mark took an interesting photo which he wants to send to Nina over the internet. The size of this picture is 5 MB. Both Mark and Nina can upload with speed 0.5 MB/s and download with speed 1 MB/s. The picture has to be fully uploaded before Nina can start downloading it, and it has to be fully downloaded before Nina can view it. How long does Nina have to wait to view the picture from the moment when Mark starts the upload?

Result. Nina has to wait at least 15 seconds.

Solution. If upload speed is 0.5 MB/s, there is 0.5 MB of data uploaded every second. In the case of Mark's picture, which size is 5 MB, is 0.5 MB tenth of its size. That means we need 10 seconds to upload it. Similarly, since Nina's download speed is 1 MB/s, the picture with size 5 MB will take 5 seconds to download. As a result, if Nina starts downloading the picture immediately after it is uploaded, she has to wait $10 + 5 = 15$ seconds.

Problem 7. Paul found a paper on the ground. There were several sentences, each of them labeled with a number.

1) Only one integer gives a product 4 when multiplied by itself.

10) Arithmetic mean of two integers is never greater than both of these numbers.

100) If an integer is divisible by four, it is also divisible by two.

1000) Each prime number is odd.

10000) If a, b, c are integers such that $a < b$ and $b < c$, then also $a < c$.

What is the sum of the labels of sentences which are true?

Result. The sum of the labels of sentences that are true is 10110.

Solution. Let us individually decide for each sentence whether it is true or false.

1) This sentence is not true because there are two such numbers, 2 and -2.

10) This sentence is true because the arithmetic mean of two numbers always lies between these two numbers (if they are distinct) or is equal to both of them (if they are the same). However, it is never greater than both of them.

100) This sentence is true because if some number is divisible by 4, it must have the number 2 in its prime factorization at least twice, so it must be divisible by 2.

1000) This sentence is not true because the number 2 is an even prime number.

10000) This sentence is true because if we start with some number, take some greater number and then take even greater number, we end up with number, which is greater than number we started with.

Consequently, sentences which are true have labels 10, 100 and 10000. The sum of these labels is 10110.

Problem 8. Albert plays with his three resistors with resistances 1Ω, 2Ω and 3Ω. What is the highest resistance he can attain using only these three resistors and a wire with infinite length and zero resistance?

Result. The highest resistance Albert can attain is 6Ω.

Solution. If we connect all resistors in series, the result will have resistance $1\Omega + 2\Omega + 3\Omega = 6\Omega$. This is also the highest resistance that we can attain from these three resistors. The reason is behind this is as follows: We show that if any pair of resistors was connected in parallel, we could increase the overall resistance by reconnecting these resistors in series. We can see it when we make a parallel connection with two sub-connections (if there are multiple sub-connections, we can treat one of these sub-connections as a single resistor, etc.). If we have total resistance in one sub-connections R_1, and in the second we have total resistance R_2, then the total resistance of this connection is

$$R = \frac{R_1 \cdot R_2}{R_1 + R_2}.$$

In case of connection in series, we have resistance $R^* = R_1 + R_2$. So, we should proof, that

$$R_1 + R_2 \geq \frac{R_1 \cdot R_2}{R_1 + R_2}.$$

However, after some transformation it is equivalent with inequality

$$R_1^2 + R_1 \cdot R_2 + R_2^2 \geq 0.$$

But, this inequality is true because each number on the left side of this inequality is positive. As a result, changing the parallel connection of resistors to the serial connection indeed increases the total resistance of the system. Thus the resistance $6\,\Omega$ is the highest resistance, which we can attain.

Problem 9. At the border crossing, there is a queue of cars. The cars leave the country with a frequency of 300 cars per hour. Cross-border worker Peter is observing the situation. On average, how often does Peter see a car leaving a country?

Result. On average, Peter can see a car every 12 seconds.

Solution. In one hour, 300 cars cross the border. This means that (on average) every minute $300/60 = 5$ cars cross the border, which is equivalent to 5 cars every 60 seconds. Therefore, on average, a car crosses the border every $60/5 = 12$ seconds. Peter can see a car leaving the country every 12 seconds on average.

Problem 10. Hannah is thinking about three numbers. The sum of these numbers is 20. The first number is equal to four times the sum of the other two numbers. The second number is seven times greater than the third number. What is the product of Hannah's numbers?

Result. The product of Hannah's numbers is 28.

Solution. Since the first number is four times the sum of other numbers, we can split all numbers into five equal parts. Four of them correspond to the first number, while the last part is a sum of the second and the third number. Thus the first number must be $20 \cdot 4/5 = 16$. The sum of the second and the third number is, therefore, $20/5 = 4$.

We also know that the second number is seven times greater than the third number. Let us split the sum of these numbers into eight equal parts. Seven of them correspond to the second number, while the last one equals the third number. Therefore, the value of the second number must be $4 \cdot 7/8 = \frac{7}{2}$ and the value of the third number is $4/8 = \frac{1}{2}$.

Consequently, the product of Hannah's numbers is $16 \cdot \frac{7}{2} \cdot \frac{1}{2} = 28$.

Problem 11. Two children play on a seesaw. Boy with mass $m_1 = 50\,\mathrm{kg}$ sits on one of seesaw's part, $r_1 = 0.6\,\mathrm{m}$ from the centre of the seesaw. Girl with mass $m_2 = 40\,\mathrm{kg}$ sits on the seesaw such that they are able to seesaw themselves. What is the distance between the boy and the girl?

Result. The girl is $1.35\,\mathrm{m}$ away from the boy.

Solution. If the girl sits on the same side of the seesaw as the boy, they would not be able to seesaw. So, they should sit on opposite sides of the seesaw. For the seesaw to work properly, the torques caused by the boy and the girl must be identical. The boy´s torque is $M_1 = m_1 g r_1$. Similarly, the girl's torque is $M_2 = m_2 g r_2$, where r_2 is her distance from the center of the seesaw. The mentioned torque equality is equivalent to

$$m_1 g r_1 = m_2 g r_2.$$

When we divide both sides of this equation by g, we can see equivalent equality

$$m_1 r_1 = m_2 r_2.$$

We should compute r_2, so we transform this equality as

$$r_2 = \frac{m_1}{m_2} r_1.$$

Distance between the girl and the center of the seesaw is $r_2 = 0.75\,\mathrm{m}$. However, on the opposite side of the seesaw, there is a boy, $r_1 = 0.6\,\mathrm{m}$ away from the center. Therefore, the distance between the girl and the boy is $r_1 + r_2 = 0.6\,\mathrm{m} + 0.75\,\mathrm{m} = 1.35\,\mathrm{m}$.

Problem 12. David wants to buy a bouquet of flowers for Veronica as a birthday present. He has 23 flower coupons. A rose costs 1 coupon, and a lily costs 2 coupons. How many different bouquets can he buy by using all of the tickets? Different bouquets are these which consist of a different number of lilies and a different number of roses.

Result. David can buy 12 different bouquets.

Solution. Notice that a rose costs 1 coupon. Because of that, every time David buys a certain number of lilies, he can use all other coupons to buy roses to spend all the coupons. Consequently, the number of different bouquets he can buy equals the number of ways he can buy lilies. The least possible number of lilies David can buy is 0; the greatest is 11 (12 is too much since it would cost 24 coupons). Therefore, the number of lilies David might buy is any number between 0 and 11 and so he can buy 12 different bouquets for Veronica.

Problem 13. Jake owns several water containers with different volumes. He has small glass with volume $0.5\,\mathrm{dl}$, glass with volume $300\,\mathrm{cm}^3$, jar with volume $0.5\,\mathrm{dl}^3$, jug with volume $1\,\mathrm{l}$ and a pot with volume $0.05\,\mathrm{hl}$. He filled all these containers with water 100 times over and poured all the water into an initially empty pool with volume $0.8\,\mathrm{m}^3$. How many extra liters of water must be poured into the pool to fill it completely?

Result. In order to fully fill the pool, we need 115 extra liters of water.

Solution. Let us first convert all the volumes to litres, remembering $1\,\mathrm{l}$ is the same as $1\,\mathrm{dm}^3$

- The small glass has volume $0.5\,\mathrm{dl} = 0.05\,\mathrm{l}$.
- The glass has volume $300\,\mathrm{cm}^3 = 0.3\,\mathrm{l}$.
- The jar has volume $0.5\,\mathrm{dm}^3 = 0.5\,\mathrm{l}$.
- The jug has volume $1\,\mathrm{l}$.
- The pot has volume $0.05\,\mathrm{hl} = 5\,\mathrm{l}$.

If we fill each of these containers once and pour it into the pool, we pour $0.05\,\mathrm{l} + 0.3\,\mathrm{l} + 0.5\,\mathrm{l} + 1\,\mathrm{l} + 5\,\mathrm{l} = 6.85\,\mathrm{l}$ of water into the pool. When we do this 100 times, we pour $100 \cdot 6.85\,\mathrm{l} = 685\,\mathrm{l}$ of water into the pool. The pool has volume $0.8\,\mathrm{m}^3 = 800\,\mathrm{l}$. In order to completely fill the pool we must pour $800\,\mathrm{l} - 685\,\mathrm{l} = 115\,\mathrm{l}$ of extra water into it.

Problem 14. Michael and Ella are spending time throwing a disc in a park. They stand 90 meters apart from each other when suddenly Ella throws the disc towards Michael with speed 36km/h such that the disc flies with constant speed over a straight line. At the exact moment, Michael starts running towards Ella with a constant speed and catches the disk in exactly 6 seconds after Ella threw it. How fast did Michael run?

Result. Michael´s speed is $5\,\mathrm{m/s} = 18\,\mathrm{km/h}$.

Solution. Let us begin with converting the speed of the disc from kilometers per hour to meters per second. Speed $36\,\mathrm{km/h}$ is the same as the speed $10\,\mathrm{m/s}$. If Michael wants to catch the disc after exactly 6s after Ella's throw, Michael and the disc must approach each other with speed $90\,\mathrm{m}/6\,\mathrm{s} = 15\,\mathrm{m/s}$. The speed at which Michael and the disc approach each other is equal to the sum of the disc's speed and Michael's speed. Consequently, that Michael's speed must be $15\,\mathrm{m/s} - 10\,\mathrm{m/s} = 5\,\mathrm{m/s}$.

Problem 15. When Matt was bored on a train, he calculated the product $1 \cdot 3 \cdot 5 \cdot \ldots \cdot 2017 \cdot 2019$. How many zero digits does Matt's product end with?

Result. Matt's product ends with no zero (ends with 0 zeros).

Solution. We will show that Matt's product does not end with a zero digit. If a given number ends with a zero digit, it is necessarily divisible by 10. Consequently, it must be divisible by two and five at the same time. However, Matt multiplies odd numbers only, and therefore, the resulting product must be odd. Hence the product is not divisible by two, so it can not end with a zero digit. Thus there is no zero at the end of Matt's product.

Problem 16. Amy drove a car from her village to a town to watch a movie in the cinema. She drove 5 times longer distance on the country road than on city road during the trip. At the same time, Amy drove 3 times as fast on the country road as on the city road. She spent 30 minutes driving on city road. How much time did she spend driving on the country road?

Result. She spent 50 minutes driving on the country road.

Solution. Amy drove for 30 minutes on the city road. If her speed were the same as on the country road (i.e., three times greater than what it was), she would have driven three times shorter, which means she would be driving only for 10 minutes. Since Amy drove 5 times longer distance on the country road, it must have taken her 5 times longer than what she would hypothetically spend driving on the city road with country road speed. Therefore, Amy had to drive 50 minutes on a country road.

Problem 17. Twelve people stood in a circle. Each of them said three statements:

1. "I live in Poland."
2. "The person on my left lives in the Czech Republic."
3. "At least one of the people standing right next to me lives in Poland, Czech Republic or Slovakia."

Then they realized that only 2 of all 36 statements were true. How many of those people live in Slovakia?

Result. Only 1 person standing in the circle lives in Slovakia.

Solution. If there would be a person from Poland standing in the circle, the first statement (s)he said would be true. At the same time, both people right next to that person would be telling the truth by saying the third statement. So at least 3 statements would be true. This means that nobody lives in Poland.

If there was a person from the Czech Republic in the circle, the person at their right would tell the true statement (the second statement). At the same time, people standing next to that person would be telling the truth by saying the third statement. Again, at least 3 of all statements would be true, which means no person lives in the Czech Republic.

If there would be nobody in the circle from Slovakia, then no statement would be true, again a contradiction.

If there was exactly one person from Slovakia standing in a circle, exactly 2 statements would be true – the second statements of the neighbors.

If there were more people from Slovakia, that would make more of the statements true, which again leads to the contradiction.

Therefore, there must be a single person from Slovakia in the circle.

Problem 18. Carl has 4 cards. Each of the cards has a number written on it. For each pair of two cards, he wrote down the sum of the numbers on them. He got three times number 14 and three times number 12. What is the sum of numbers on all the cards?

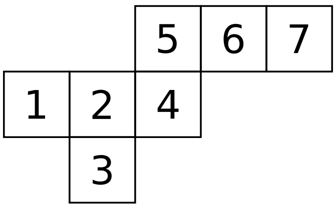
Result. The sum of all numbers on Carl's cards is 26.

Solution. Let us calculate the sum of the sums of all pairs. This sum is $3 \cdot 14 + 3 \cdot 12 = 78$. Each card number was added exactly three times to that sum as it made a pair with each of 3 other cards. Because of this, the sum of the sum of all pairs is three times bigger than the sum of all cards. Therefore the sum of Carl's cards is $78/3 = 26$.

Problem 19. George cut a piece of paper to create a cube surface, as the picture below shows. However, the piece of paper has one extra square that it should not have. Therefore George wants to cut out one square such that the piece of paper

1. remains connected, and
2. can be folded to form a cube surface.

Find all possible squares he can cut off.



Result. George may cut off any of the squares labeled 3 and 7. It is necessary to find both solutions.

Solution. If we removed any squares labeled 2, 4, 5, or 6, the surface would not be connected anymore. Therefore we can potentially cut off only the squares labeled 1, 3 or 7. If we tried to form a cube from the surface, we would notice that the square labeled 1 has to be opposite the square labeled 4 and the square labeled 2 has to be opposite the square labeled 6. More interestingly, the square labeled 7 has to be opposite the square labeled 5. Simultaneously, 5 must be opposite the square labeled 3. Since there can be only one square opposite the square labeled 5 we have to cut off one of the squares labeled 3 and 7. Notice that if we cut off any of them, we will be able to fold the paper to form a cube surface. Therefore we can cut off any of the squares labeled 3 and 7.

Problem 20. Scientists have performed the following experiment. They took a hollow iron cube and cut out a hole of area $10\,\mathrm{cm}^2$ from one side. The cut-out piece was replaced with a cover of identical shape. Then they sucked all the air from inside the cube so that there was a vacuum inside the cube while keeping the air outside of the cube. Next, they turned the cube such that the covered hole faces down. What is the maximal possible mass of the cover so that the cover does not fall?

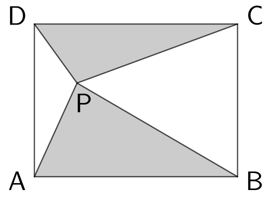
Result. The maximal possible mass of the cover is 10 kg.

Solution. Let us look at the forces that have an impact on the cover. The force acting downwards is the gravity $F_G = mg$, where m is the mass of the cover, and g is the gravitational acceleration. Around the cube, air with atmospheric pressure p_A acts on the weight with pressure force $F_T = Sp_A$. Inside the cube is a vacuum, whose pressure is zero, so it does not act on the weight. If the weight does not move, the magnitude of pressure force F_T must be at least as big as the magnitude of the gravity F_G. So we must have $F_T \geq F_G$. Plugging in the formulas to calculate these forces we get $Sp_A \geq mg$, which is equivalent to $m \leq Sp_a/g$. The maximal mass of the cover is 10 kg.

Problem 21. Laura drew a rectangle $ABCD$ such that the ratio between lengths of the sides AB and BC was $4:3$. Then she drew a point P inside the rectangle. She noticed that the areas of triangle ABP, BCP and CDP were $17\,\mathrm{cm}^2$, $42\,\mathrm{cm}^2$ a $47\,\mathrm{cm}^2$ respectively. What is the area of triangle DAP?

Result. The area of triangle DAP is $22\,\mathrm{cm}^2$.

Solution. Each of the triangles ABP a CDP shares a common side with the longer side of rectangle $ABCD$. The altitude to this side represents the distance between P and this side. Moreover, the sum of the distances from the longer rectangle sides to the point P is equal to the length of the shorter side of the rectangle. We may express the sum of the areas of triangles ABP and CDP from the problem statement as $17\,\mathrm{cm}^2 + 47\,\mathrm{cm}^2 = 64\,\mathrm{cm}^2$. On the other hand, this sum of the areas must equal to the half of the length of longer side of rectangle multiplied by the length of the shorter side of this rectangle. Hence the sum of these two triangles equals to the half of the area of the rectangle $ABCD$. So we may deduce that also triangles BCP and DAP have combined area $64\,\mathrm{cm}^2$. Thus the area of triangle DAP must be $64\,\mathrm{cm}^2 - 42\,\mathrm{cm}^2 = 22\,\mathrm{cm}^2$.



Problem 22. Dan bought a water pump in order to fill his pool. In the manual, he learned that the volumetric flow rate of the pump is $2\,\mathrm{dm}^3/\mathrm{s}$. Dan's pool is $9\,\mathrm{m}$ long, $3\,\mathrm{m}$ wide and $2\,\mathrm{m}$ deep. How long does it take to fill the empty pool with his pump completely?

Result. Filling of the pool with the pump takes 27000 seconds = 450 minutes = 7.5 hours.

Solution. The fact that pump can pump water with volumetric flow rate $2\,\mathrm{dm}^3/\mathrm{s}$ means, that it can pump water with volume $2\,\mathrm{dm}^3 = 0.002\,\mathrm{m}^3$ every second. Dan wants to fill the pool with volume $9\,\mathrm{m} \cdot 3\,\mathrm{m} \cdot 2\,\mathrm{m} = 54\,\mathrm{m}^3$. Time required to fill pool with volume $54\,\mathrm{m}^3$ is

$$\frac{54\,\mathrm{m}^3}{0.002\,\mathrm{m}^3/\mathrm{s}} = 27000\,\mathrm{s}.$$

Problem 23. In Naboj country there are strange blocks of flats. They have 9 floors each. The staircase from the ground floor to the 1st floor consists of a certain number of steps. Each following staircase has one more step. The number of steps that one needs to climb from the ground floor to the 5th floor is the same as the number of steps that one needs to climb from the 5th floor to the 9th floor. How many steps does the staircase from the ground floor to the 1st floor consist of?

Result. The staircase from the ground floor to the 1st floor has 16 stairs.

Solution. Let us form a particular pairing of staircases. We match 2nd (i.e., staircase between the first and second floor) and 6th staircases, 3rd and 7th, 4th and 8th, and 5th and 9th staircases. Given the pairing, the set of staircases which are first in pair form a path from 1st to 5th floor. On the other hand, a set of staircases, which are second in pairs, form a path from 5th to 9th floor.

Notice that the second staircase in each pair has 4 more steps than the first staircase in this pair. Therefore, the path from 1st to 5th floor must have $4 \cdot 4 = 16$ fewer steps than the path from 5th floor to 9th floor. However, it is given that the path from the ground to 5th floor has the same number of steps as the path from 5th floor to 9th floor. We have already established that the path from the ground to the 5th floor has 16 more steps than the path from 1st to 5th floor. Therefore, the staircase from the ground floor to 1st floor must have 16 steps.

Problem 24. Jack found that protons and neutrons are made of two types of quarks - quarks of type *up* and quarks of type *down*. A proton contains two quarks of type *up* and one quark of type *down*. Neutron contains one quark of type *up* and two quarks of type *down*. Jack learned that the electric charge of proton and neutron must be equal to the sum of electric charges of quarks they contain. What is the electric charge of a particle that contains one quark of type *up* and one quark of type *down*?

Note: Submit your result as a multiple of elementary electric charge $\mathbf{e}$. Proton is a particle with electric charge $+\mathbf{e}$ and neutron is a particle without electric charge.

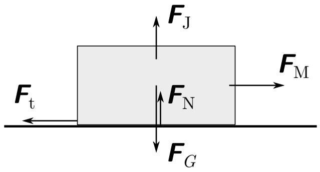
Result. The electric charge of such particle would be $1/3\,\mathbf{e}$.

Solution. The sum of electric charges of one proton and one neutron is $\mathbf{e}$. In total, they contain 3 quarks of type *up* and 3 quarks of type *down*. To get a "particle", which would contain only one quark of type *up* and one quark of type *down*, we need to reduce the number of both types of quarks to one third. This will also reduce the total electric charge to one third. So Jack's particle would have electric charge $1/3\,\mathbf{e}$.

Problem 25. Jonas and Michelle put a cube with mass 2 kg on the table. The coefficient of friction between the cube and the table was 0.6. Jonas pulled the cube with force 5 N in the vertical direction. Michelle pulled the cube in a horizontal direction such that the cube moved with constant velocity. Calculate Michelle's force.

Result. Michelle pulled the cube with force with size 9 N.

Solution. Let us examine the forces acting on the cube.



In the horizontal direction, we have Michelle's force F_M and the force of friction F_t. These two forces must have the equal magnitude to make the cube move with constant velocity. At this moment, we can't calculate the magnitude of F_t. In the vertical direction, we have gravitational force F_G and Jonas's force $F_J = 5\,\mathrm{N}$. These two forces don't have and equal magnitude (obviously $F_G > F_J$). The cube must therefore act on the table with a certain force. From Newton's third law we have, that the table must act on the cube with a force of the same magnitude but opposite direction. Denote this force by F_N. The cube doesn't move in the vertical direction; thus the sum of forces acting on it in the vertical direction must be zero. Hence we have $F_N = F_G - F_J$. Using this force, we can calculate the magnitude of the friction force: it depends only on F_N and the coefficient of friction $f = 0.6$. The formula we should use is $F_t = fF_N$. Putting Michelle's force in equality with the friction F_t we get

$$F_M = F_t = f(F_G - F_J) = f(mg - F_J) = 0.6 \cdot (2\,\mathrm{kg} \cdot 10\,\mathrm{N/kg} - 5\,\mathrm{N}) = 9\,\mathrm{N}.$$

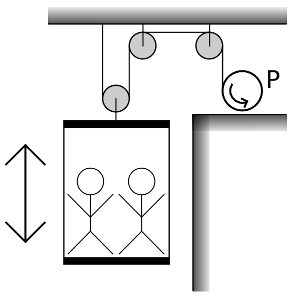
Problem 26. Each of three friends – Albert, Bernard, and Cyril – has a certain number in mind. The least common multiple of Albert's and Bernard's number is 24. The least common multiple of Bernard's and Cyril's number is 40. What is the smallest possible value of the least common multiple of Albert's and Cyril's number?

Result. The smallest possible value of the least common multiple of Albert's and Cyril's number is 15.

Solution. The least common multiple of Albert's and Bernard's numbers is divisible by 3, but the least common multiple of Cyril's and Bernard's is not. This means that Bernard's number is not divisible by 3, so Albert's number has to be divisible by 3. Analogically, Cyprian's number must be divisible by 5.

The least common multiple of Albert's and Cyprian's number must be at least $3 \cdot 5 = 15$. This value is reachable when Albert thinks of 3, Bernard of 8, and Cyril of 5. It is also easy to prove that these three numbers meet the given conditions. The least possible value of the least common multiple of Albert's, Bernard's, and Cyril's number is 15.

Problem 27. There is a new prototype of an elevator at the school that works. The main idea of how the elevator works are sketched in the picture below. The engine has power $P = 1\,\mathrm{kW}$. What is the maximal speed of the elevator, while ascending, if the mass of the elevator is $m = 400\,\mathrm{kg}$?



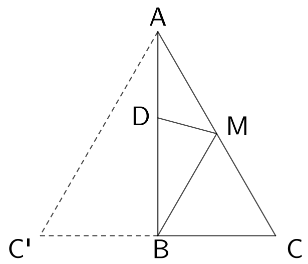
Result. Maximal speed of this elevator is $0.25\,\mathrm{m/s} = 0.9\,\mathrm{km/h}$.

Solution. Notice that it does not matter what the principle of working of our elevator is. The only thing that matters is, that elevator transforms energy from the engine to the potential energy of the elevator. When the elevator's height increases by Δh, its potential energy increases by $\Delta E = mg\Delta h$. Moreover, if that change is done in time Δt, the engine has power $P = \Delta E/\Delta t = mg\Delta h/\Delta t$. Notice that expression $\Delta h/\Delta t$ is speed v. This is the speed of the cabin's motion. If we transform this equation, the result is $v = P/mg$. The maximal speed of elevator cabin is $v = 0.25\,\mathrm{m/s}$.

Problem 28. Patrick drew a right triangle ABC with a right angle at vertex B. The size of the angle BAC was $30°$. Then he marked a midpoint of AC and named it M. Finally, he added a point D on side AB such that $|BD| = |BC|$. Find the size of the angle BMD.

Result. The size of the angle BMD is $75°$.

Solution. Triangle ABC has angles of sizes $90°$, $30°$ and $180° - 90° - 30° = 60°$. Suppose we take two such triangles and glue them together alongside side AB. The resulting object is an equilateral triangle since all of its angles are $60°$. Looking back at the original triangle ABC, we see that hypotenuse of ABC is a side of the equilateral triangle, and the side opposite the angle $30°$ is half of the side of large equilateral triangle. Therefore in the triangle ABC, we get $|AC| = 2 \cdot |BC|$.



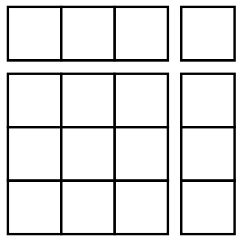
Since M is the midpoint of AC, we see that $|CM| = |AC|/2 = |BC|$. Thus BCM is an isosceles triangle with base BM. Since the angle BCM is $60°$, we can easily show that also other two angles in the triangle BCM are $60°$ as well. Therefore, BCM is equilateral triangle, hence $|BC| = |BM| = |CM|$. However, we also have $|AM| = |CM| = |BM|$, and therefore ABM is an isosceles triangle with base AB. From this, we see that the size of the angle ABM equals the size of the angle BAM, which is $30°$.

Finally, we know that $|BD| = |BC|$, which also equals $|BM|$ as we showed. So BDM is an isosceles triangle as well, with a base BM. We know that the size of the angle DBM is $30°$. Both of the remaining angles in this isosceles triangle have the size $(180° - 30°)/2 = 75°$. Thus the size of the angle BMD is $75°$.

Problem 29. Dwarfs in Naboj country decided to tile the city square. In the beginning, the first dwarf places the first tile into a corner of the square. Then the second dwarf places new tiles into the same corner so that all the tiles now form a 2×2 square. The third dwarf adds new tiles so that the size of the tiled square again increases by one. Following dwarfs continued enlarging the square in the same manner. How many tiles does the 2019-th dwarf place?

Result. The 2019-th dwarf placed 4037 tiles.

Solution. Let us study the pattern in which each dwarf places new tiles. For instance, the picture demonstrates how the fourth dwarf places his tiles. He attaches new tiles to two sides of the currently tiled square. He adds as many tiles to each side as the size of the previous square, and then he adds one extra tile to fill the corner. Therefore the fourth dwarf places $3 + 3 + 1 = 7$ tiles, as we can see. Analogously, every following dwarf has to attach tiles to both sides of the previous square and add one extra. Thus 2019-th dwarf has to place $2018 + 2018 + 1 = 4037$ tiles.



Problem 30. Mike dropped a ball from height 10 m. Every time the ball touches the floor it halves its speed. How many times will the ball bounce at least 1 cm over the ground?

Result. The number of bounces that are at least 1 cm over the ground is 4.

Solution. When the ball is bouncing, its energy continuously changes from potential to kinetic energy and vice versa. If we drop the ball from some height, then all of its energy at the beginning will be potential energy. A very short moment before it bounces is all the energy transformed to kinetic energy. When it bounces, it loses some kinetic energy and then it bounces back in the air. Finally, it reaches a new maximal height, where all of its energy is potential, and the whole process may repeat.

The formula to calculate the kinetic energy E_k is $E_k = mv^2/2$, where v is the ball's speed and m is the mass. Therefore, halving the speed of the ball results in the value of kinetic energy decreases by the factor if one-fourth. Reducing the kinetic energy to one-fourth will also reduce the potential energy at the maximal height to one-fourth. The formula to calculate potential energy E_p is $E_p = mgh$, where m is the ball's mass, g is the gravitational acceleration and h is the height of the ball. Thus if we reduce the potential energy to one-fourth, the ball will bounce into four times lesser height.

Therefore it suffices to calculate how many times we can divide 10 m by four to still obtain a height in centimeters, whose quantity will be greater than 1. If we convert 10 m to centimeters, we get $10\,\mathrm{m} = 1000\,\mathrm{cm}$. After the fourth bounce the ball will bounce into the height $1000\,\mathrm{cm}/4^4 = 1000\,\mathrm{cm}/256 > 1\,\mathrm{cm}$. After the fifth bounce, it will bounce into the height $1000\,\mathrm{cm}/4^5 = 1000\,\mathrm{cm}/1024 < 1\,\mathrm{cm}$. Therefore the ball will bounce into the height greater than 1 cm exactly 4 times.

Problem 31. Witcher Bob put an empty glass on the water such that 3/5 of its volume were under water. After some time, he put the same glass into a magic potion, where 3/4 of its volume was under that potion. What is the density of this potion?

Result. The density of magic potion is $800\,\mathrm{kg/m^3}$.

Solution. If we take some object with density ρ_1, and volume V, and we let it flow on the liquid with density ρ_2 ($\rho_2 > \rho_1$), the volume of this object, which is under water (V') is

$$V' = \frac{\rho_1}{\rho_2} V.$$

So the density of the glass is 3/5 of the density of water, which is $600\,\mathrm{kg/m^3}$. This density is also 3/4 of the density of magic poison. As a result, density of this poison is $800\,\mathrm{kg/m^3}$.

Problem 32. Aliens Jake and Jane live on a planet that is orbiting around the central star clockwise and rotating around its axis clockwise. An "alien day" means the period between two "sunsets" of the central star. Every alien day is 6 hours long, and it takes 35 alien days until the planet makes a complete orbit around the central star. How long does it take for the planet to rotate around its axis?

Result. One rotation around the planet's axis takes 5 hours and 50 minutes = 350 minutes.

Solution. It might seem that one rotation around the axis takes exactly one alien day. Nonetheless, this is not the case. Let us pick a point on the planet at the very moment when the central star rises. One rotation around the axis later, the planet will be rotated exactly the same regarding space around it. However, it has changed its position with respect to the central star. Because the planet both rotates and orbits clockwise, at this time, the point will be still before the rise of the star, so there is a little shift. When we look at those little shifts during one orbit around the star, they sum up in such a way that it takes 36 rotations around the axis for 35 alien days to pass. Another way how to make this observation is like this: The planet makes 36 rotations around the axis, but the observer placed on the picked point sees the star rising 1 time less due to these shifts, so only 35 times. 35 alien days take $35 \cdot 6 \cdot 60 = 12600$ Earth minutes. Therefore one rotation takes $12600/36 = 350$ Earth minutes.

Problem 33. When Hannah was bored on a train, she calculated the product $2019 \cdot 2019 \cdot 2019 \cdot 2019$. Then she summed the digits of her product. Next, Hannah obtained a new number and summed its digits. She continued summing the digits until she obtained a one-digit number. Which number did she obtain?

Result. Hannah obtained a number 9.

Solution. Notice that the number 2019 is divisible by three ($2019 = 3 \cdot 673$). So the product of four numbers 2019 must be divisible by nine. If some number is divisible by nine, its sum of digits is also divisible by nine. If the sum of digits of a number is divisible by nine, the sum of digits of the sum of digits of this number must be divisible by nine as well. We could continue reasoning this way. But this means, that the resulting one-digit number must also be divisible by nine. There are two one-digit numbers divisible by nine - 0 and 9. However, the only number whose sum of digits is 0 is 0 itself, so Hannah obtained the number 9.

Problem 34. The king gave Archimedes a task. His goal is to find out whether the new king's crown is made of pure gold or if there is some silver inside. Archimedes immersed the crown into the water and realized that its volume is $V = 0.14\,\mathrm{l}$. He did the same thing with pure gold and pure silver. Pure gold with the mass equaling to the mass of crown had a volume $V_{Au} = 0.11\,\mathrm{l}$ and pure silver with the same mass had a volume $V_{Ag} = 0.2\,\mathrm{l}$. Therefore he said that the crown had not been made of pure gold. What part of the crown's mass is made up of a mass of gold?

Result. The mass of gold makes up $2/3 \doteq 66.67\%$ of the crown's mass.

Solution. Let the mass of gold makes up p of the mass of the crown, where p is the number between 0 a 1. Mass of silver then makes $1 - p$ of the mass of the crown. Since gold makes up p of the mass of the crown, the volume of the gold in the crown is pV_{Au}. Similarly, the volume of silver in the crown is $(1 - p)V_{Ag}$. The sum of these volumes should be equal to the volume of the crown. Thus we have

$$V = pV_{Au} + (1 - p)V_{Ag}.$$

By solving this for p we get

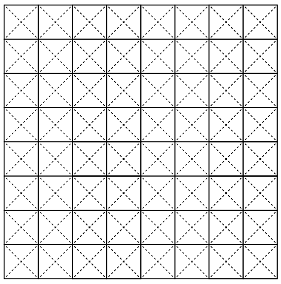
$$p = \frac{V_{Ag} - V}{V_{Ag} - V_{Au}}.$$

Substituting the known values into the above formula, we get $p = 2/3$. As a result, the crown's mass is made up of 2/3 mass of gold.

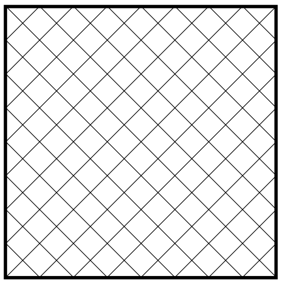
Problem 35. Teresa has a square paper measuring 8×8 centimeters. She folds it twice and gets a square 4×4 centimeters. Teresa repeats the same procedure twice more until she obtains a square of 1×1 centimeters. Then she cuts both diagonals of the square. How many different pieces of paper will she have?

Result. Teresa will get 144 pieces of paper.

Solution. If we did not cut the paper in the end, the lines where we folded the paper would form an entire 8×8 square grid. Cutting the paper gets us both diagonals cut out in every square of this grid. We can draw these cuts into an image:

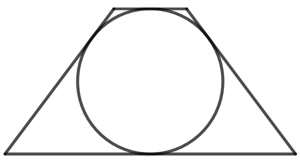


To make new pieces of paper more visible, let us draw one more picture, where we omit the folding lines forming the grid:



It is easy to see we get only two different shapes - squares inside the paper and triangles on its perimeter. We can notice that every square and triangle contains precisely one edge of the original grid and also, every edge is contained in exactly one piece of paper. Therefore, the total number of different pieces must be equal to the number of edges of the grid. We can calculate that there are $8 \cdot 9 = 72$ edges in each direction, therefore $2 \cdot 72 = 144$ edges in total. Consequently, Teresa is going to have 144 pieces of paper.

Problem 36. Laura drew an isosceles trapezoid whose legs were 5 centimeters long. She realized that she could inscribe a circle into this trapezoid with a radius of 2 centimeters. What was the area of Laura's trapezoid?



Result. The area of Laura's trapezoid was $20\,\mathrm{cm}^2$.

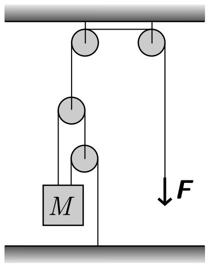
Solution. Notice that it is not possible to inscribe a circle into any quadrilateral. Instead, the quadrilateral must satisfy a rather particular property. Consider any quadrilateral to which one can inscribe a circle. Let us mark all points where the inscribed circle touches the quadrilateral – there is exactly one on each side. At this moment, every side of the quadrilateral is divided into two line segments, while each of them is tangent to the inscribed circle. We have two such line segments for each vertex of the quadrilateral, each of which has one of the vertices as an endpoint. If we drew a picture, which would consist of only these two line segments and the inscribed circle, it would be symmetrical by the angle bisector of the angle formed by these two line segments. Therefore these two line segments must be of an equal length. Let us now focus on a group of four line segments that belong to the sides facing each other. There are two such groups. Interestingly, every line segment from the first group can be matched with a line segment from the second group of identical length. Consequently, the sum of lengths of quadrilateral sides that do not face each other must be a constant.

Let us come back to Laura's isosceles trapezoid. We know that both legs of the trapezoid are 5 cm long. Since a circle can be inscribed into this trapezoid, the sum of lengths of two other sides (bases) must be equal to the sum of lengths of the legs. Therefore, the sum of lengths of the bases must be 10 cm. This is exactly what we needed – the sum of lengths of bases appears in the formula to calculate the trapezoid area.

It remains to determine the height of the trapezoid. We know that the line segment connecting the center of the inscribed circle with the point of tangency of one of the sides is perpendicular to this side. Since the bases of the trapezoid are parallel, those two line segments must be parallel as well. Moreover, since both line segments pass through the center of the inscribed circle, they must lie on the same line perpendicular to both bases. Their sum of the lengths must therefore be the trapezoid height, which in turn must also equal the circle diameter. This means that the diameter of the inscribed circle is the same as the height of Laura's trapezoid.

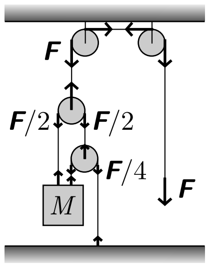
In summary, the sum of lengths of bases of Laura's trapezoid is 10 cm and its height is 4 cm. Consequently, the area of this trapezoid is $10\,\mathrm{cm} \cdot 4\,\mathrm{cm}/2 = 20\,\mathrm{cm}^2$.

Problem 37. Bob the builder wants to make work at construction site easier. Dizzy advised him to build a pulley system as sketched in the picture below. Bob got excited and immediately used the system to lift a 15-kilogram box. What is the least amount of force Bob needs to pull the rope with so he can lift the box?



Result. Bob needs to pull with force at least 200 N.

Solution.



If Bob wants to lift the box using this system of pulleys, he must pull the rope with the same force he needs to keep the box at rest. In this case we see that no rope is moving, no pulley is rotating and no movable pulley is moving. So for each part of the rope we must have two forces with the same size but with opposite directions acting on it. The force F that Bob's pulls the rope will move to the right fixed pulley. Momentums of forces acting on the pulley must be equal, so the force F will move to the left fixed pulley. Because of it, the force F will move to the upper movable pulley. To get the equal momentums of forces acting on this pulley we must have that the other two forces acting on this pulley must have equal size. This movable pulley is not moving, so the force which pulls it upwards must be the same as the sum of the forces which pull it downwards. Hence both of these forces have the size $F/2$. One of the ropes from this pulley will move the force $F/2$ directly to the box. The other on will move it to the lower movable pulley. Also here will the force divide into two forces of equal sizes. Consequently each rope will act on this pulley with force $F/4$. Only one of these forces will move to the box. Thus the sum of the forces acting on the box will be $F/2 + F/4 + 3F/4$. To make this box be at rest, this force must have equal size as the gravitational force acting on this box. So we can express the force Bob needs to use

$$F = 4Mg/3.$$

Bob needs to pull the rope with force at least 200 N.

Problem 38. Blake took two of his favorite positive integers. He realized that the product of these numbers is 7-times greater compared to their sum. What is the sum of Blake's two favorite numbers?

Result. The sum of Blake's favorite numbers is 64.

Solution. Denote by x and y Blake's favorite numbers. It does not matter which is greater, so let us say $x > y$ without loss of generality. From the problem statement, we have

$$x \cdot y = 7 \cdot (x + y).$$

When we collect all terms on one side of the equation, we get

$$x \cdot y - 7 \cdot x - 7 \cdot y = 0.$$

Our goal is to write the above equation as a product of two parentheses such that each parenthesis contains exactly one of the numbers x and y. Having that in mind, and noticing that the left side of the equation looks similar to the product $(x - 7) \cdot (y - 7)$ we might get an idea of what to do. In particular, let us add 49 to both sides of equation $x \cdot y - 7 \cdot x - 7 \cdot y = 0$. We get

$$(x - 7) \cdot (y - 7) = 49.$$

If one of the parenthesis were negative, the other one would also have to be negative. Numbers x and y are positive integers so in this case the values of the parentheses could be $-1, -2, -3, -4, -5$ and -6. So the maximal value of the product $(x - 7) \cdot (y - 7)$ would be at most $(-6) \cdot (-6) = 36$, which is less than 49.

Hence the value of both parentheses must be positive. Moreover, their values are integers and thus must be divisors of the number 49. This number can be factorized in two ways: $49 = 49 \cdot 1 = 7 \cdot 7$. If the second option was used, we would have $x = y = 14$. But these numbers have to be distinct. Consequently, one of the parenthesis must be equal to 49 and the other one 1. At the beginning we assumed $x > y$ so $x - 7 = 49$ a $y - 7 = 1$. This means, that $x = 56$ and $y = 8$. This is the only solution; therefore the sum of Blake's favorite numbers is 64.

Problem 39. Dan is a homogeneous cube with side length $a = 2\,\mathrm{m}$ and density $\rho = 800\,\mathrm{kg/m^3}$. He is falling from the height of $h = 6\,\mathrm{m}$ measured from his bottom face into a lake. What is the maximum depth that his top face will dive into?

Note: Assume that the water in the lake is an ideal liquid.

Result. The maximal depth that Dan's top face will dive into is $27\,\mathrm{m}$.

Solution. Denote by d the maximal depth in which Dan's top face dove. So his bottom face dove into depth $d + a$. His bottom face then traveled $h + d + a$ downwards while falling, and thus his potential energy must have been reduced by energy $mg(h + d + a)$, where g is the gravitational acceleration and m is Dan's mass, which can be expressed as $m = V\rho = a^3\rho$.

Let us examine where did this energy disappear. At the place of Dan's maximal depth, there used to be water. But now that Dan's there, the water must have moved somewhere else. The water is incompressible so it must have moved upwards to the surface of the lake. This water forms some cuboid on the whole surface of the water. The area of its base is the same as the area of the lake (i.e., huge) and therefore, the height of the cuboid must be extremely small. Let us omit it for the sake of simplicity and suppose that all of the water has spread across the surface of the lake.

Hence we can calculate the change of the potential energy of water. This water initially formed a cube with the same side length as Dan's side. Thus its mass is $m = a^3\rho_{water}$. To make calculations easier suppose that all of the cube's mass is in its center of gravity. The center of gravity of the cube was initially in depth $d + a/2$. This means that its potential energy has increased by energy $a^3\rho_{water}g(d + a/2)$. We have two energies transforming one into another. Let us put them into equality:

$$a^3\rho g(h + d + a) = a^3\rho_{water}g(d + \frac{a}{2}).$$

After dividing by a^3 and g we get

$$\rho(h + d + a) = \rho_{water}(d + \frac{a}{2}).$$

We may express the depth d

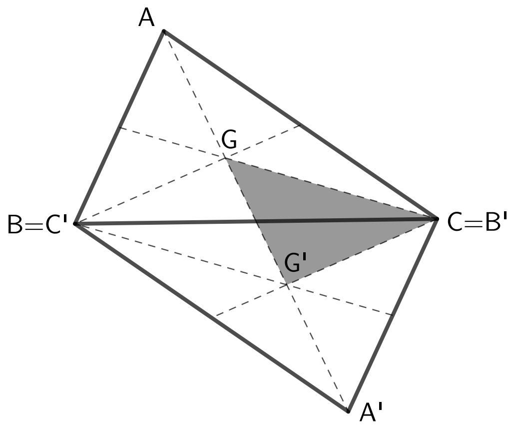
$$d = \frac{\rho(h + a) - \rho_{water}\frac{a}{2}}{\rho_{water} - \rho}.$$

Inserting the known values, we get that the maximal depth, in which Dan's top face can dive, is $27\,\mathrm{m}$. In reality, Dan might not dive so deep. Part of his potential energy would transform due to the drag, waves or into the energy of the water splash. Consequently, the resulting depth is going to be lower. However, in the ideal scenario where the above issues are non-factor, Dan is going dive into the depth $27\,\mathrm{m}$.

Problem 40. Tom drew a triangle ABC, with the medians of lengths 6, 8, and 10 centimeters. Calculate the area of this triangle.

Result. The area of Tom's triangle is $32\,\mathrm{cm}^2$.

Solution. The medians of the triangle intersect at the centre of gravity T, which divides each of the medians in ratio $2:1$ such that the longer part is at the vertex. Consequently, the distance between T and the side of the triangle equals one-third the length of the corresponding altitude.



Take the triangle whose one vertex is T, and the other two vertices are two vertices of triangle ABC, e.g., triangle ABT. The altitude of this triangle to side AB has a length of $v_c/3$ and thus the area of triangle ABT is $S_{ABT} = c(v_c/3)/2 = S/3$, where $S = cv_c/2$ is the area of triangle ABC that we need to find.

Moreover, one of the medians passes through the triangle ABT, dividing it into two smaller triangles. These triangles have the same length of altitude $v_c/3$ and the same length of base $c/2$. Hence they both have area $S_{ABT}/2 = S/6$. Applying the analogous thought process to triangles BCT and ACT, we deduce that the medians divide the triangle ABC into 6 parts if identical area.

Next, we perform a trick to make our life easier: we invert the whole figure through the midpoints of each side. Let us invert triangle ABC through the midpoint of side AB, as in the picture. In the new picture, take the triangle $TT'B$ (colored), consisting of 2 triangles formed by the medians. So its area is $S_{TT'B} = 2 \cdot S/6 = S/3$. Furthermore, the lengths of its sides are $2/3$ of lengths of the medians, so $4\,\mathrm{cm}$, $16/3\,\mathrm{cm}$ and $20/3\,\mathrm{cm}$.

To calculate its area more simply, we may notice that the lengths of the medians satisfy the Pythagorean theorem (so does the triangle rescaled by a factor of $2/3$). That means the triangle $TT'B$ is right-angled with legs of lengths $4\,\mathrm{cm}$ and $16/3\,\mathrm{cm}$. Therefore, the area of $TT'B$ is

$$S_{TT'B} = \frac{4\,\mathrm{cm} \cdot \frac{16}{3}\,\mathrm{cm}}{2} = \frac{32}{3}\,\mathrm{cm}^2 . \implies S = 3S_{TT'B} = 32\,\mathrm{cm}^2$$

Therefore the area of Laura's triangle is $32\,\mathrm{cm}^2$.

Problem 41. Andy and Mary are running on a circular track. Andy runs faster. They started running from the same place in the same direction and met again after $t_1 = 420$ seconds. Then they started running again but in the opposite direction. They met after $t_2 = 70$ seconds. How many more seconds does Mary require to run a lap compared to Andy?

Result. Mary needs 48 more seconds to run a lap.

Solution. Denote by T_A the time Andy needs to run a lap, T_M the time Mary needs to run a lap and s the length of a lap. From this we can derive Andy's speed as $v_A = s/T_A$ and Mary's speed as $v_M = s/T_M$. When they run with the same direction, their relatively (with respect to each other) speed v_1 equals the difference of their individual velocities, hence $v_1 = v_A - v_M$. The relative speed v_1 can be also expressed as $v_1 = s/t_1$. By comparing these two relations we have

$$\frac{s}{t_1} = \frac{s}{T_A} - \frac{s}{T_M}.$$

Dividing the whole equation by s we get

$$\frac{1}{t_1} = \frac{1}{T_A} - \frac{1}{T_M}.$$

When Andy and Mary run in the opposite direction, their relative speed v_2 is the sum of their velocities, i.e., $v_2 = v_A + v_M$. This relative speed can be alternatively expressed as $v_2 = s/t_2$. By comparing these two equations and after dividing by s we get

$$\frac{1}{t_2} = \frac{1}{T_A} + \frac{1}{T_M}.$$

We got two equations with two variables T_A and T_M; those are the values we need to calculate. By summing of these equations we obtain

$$\frac{1}{t_1} + \frac{1}{t_2} = \frac{2}{T_A},$$

which is equivalent to

$$T_A = \frac{2 \cdot t_1 \cdot t_2}{t_1 + t_2}.$$

Similarly we could subtract the equations instead of summing them, in which case we would arrive at

$$T_M = \frac{2 \cdot t_1 \cdot t_2}{t_1 - t_2}.$$

It is easy to plug in values we know to get $T_A = 120$ seconds and $T_M = 168$ seconds. From this, we immediately have $T_M - T_A = 48$ seconds, which means that Mary needs 48 more seconds to run a lap compared to Andy.

Problem 42. Thirteen players participated in the chess tournament. Each of them played against each other exactly once. After the tournament, they realized, that each of the players won exactly 6 of the matches and lost exactly 6 of the matches. None of the matches ended up a draw. How many possible ways can one choose a group of 3 players such that each has exactly one victory and one defeat against the other players within the group?

Result. There are 91 ways to choose a group of 3 players satisfying the given conditions.

Solution. Let us examine how the groups of three players might look like; in particular, look at the matches within the group members. In the first group type, each player won exactly once (we should figure out the number of such groups). In the second group type, there is a player who won twice, a player who won once, and a player who did not win.

The above two cases correspond to the only two possible match outcomes within the group since the players in each group have combined three victories from the matches between them. Therefore, to calculate the number of triplets/groups of the first type, we can calculate the number of all groups and then subtract the number of groups of the second type from it.

Let us begin with the simpler part and calculate the number of all groups. When choosing a group of players, we choose the first player among 13 players, the second among 12 players, and the third among 11 players. Overall we have $13 \cdot 12 \cdot 11$ ways to choose this group. However, each of the groups was counted more than once. Specifically, each group was counted for every single ordering of the players within the group. Hence we have to divide the obtained number by the number of ways to order players in the group. We choose the first player among all 3 players, the second among 2 players, and the third among a single player. We can order them in $3 \cdot 2 \cdot 1 = 6$ ways, so the number of all groups of three players must be $13 \cdot 12 \cdot 11/6 = 286$.

Next, we need to calculate the number of groups where someone won twice. As we want to count all the groups where someone won twice, we can precisely name the player that won both matches for every group. We can reverse this idea. For each player, we can find all the groups where (s)he won twice and count the number of them. Summing this number through all the players, we obtain the count of all groups where someone won twice. Note that we counted all such groups and did not count any of the groups more than once.

For a given player, let us count the number of triplets that (s)he won all the matches within. To "build" such a triple, we have to choose two players among the six players (s)he has defeated. We have 6 ways to choose the first and 5 ways to choose the second. Since we do not want to take their order into account, we must divide this number by 2. We get that for each player there is $6 \cdot 5/2 = 15$ groups where (s)he won both matches. Summing this throughout all players, we learn that the number of groups of the second type is $13 \cdot 15 = 195$.

Finally, we have to determine the number of groups where everyone won exactly once. We already know that all the triplets can be divided into groups where everyone won once and the groups where somebody won twice. Therefore the number of groups in which everyone won exactly once is $286 - 195 = 91$.