# Solutions

9th Náboj Junior

19th November 2021

# Acknowledgements

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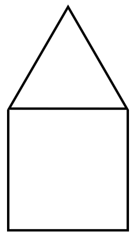
**Problem 1.** Matej got three scoops of ice cream. The first scoop costs 0.80 €; the second one costs 0.70 € while the third one costs 1.05 €. Matej paid for them with 5 € banknote. How many euros did he get back from the ice cream man?

*Result.* 2.45

*Solution.* Matej paid with 5 € banknote for three ice creams, whose total price was $0.80\,€ + 0.70\,€ + 1.05\,€ = 2.55\,€$. So the ice cream man gave Matej $5\,€ - 2.55\,€ = 2.45\,€$ back.

**Problem 2.** Naty wants to draw a house - that is, a square with an equilateral triangle as a roof (see figure). She wants the side length of the square to be 0.5 m. It takes Naty one second to draw 1 dm of line. How many seconds does it take her to draw the whole picture of a house?



*Result.* 30

*Solution.* All four sides of the square part are the same length, 0.5 m. All sides of the triangular roof have are of a mutually identical length, too. Note that one of the lines is a side of both the square and the triangle. Hence all lines in the picture must have the same length, that is, 0.5 m. The picture consists of six such lines and thus the sum of all of their lengths is $6 \cdot 0.5\,\mathrm{m} = 3\,\mathrm{m}$. Every second, Naty draws 1 dm of line, therefore for every 10 second elapsed, she draws a line with a total length of $10 \cdot 1\,\mathrm{dm} = 10\,\mathrm{dm} = 1\,\mathrm{m}$. To draw the full picture, she has to draw three times as much, so the drawing will take her $3 \cdot 10\,\mathrm{s} = 30\,\mathrm{s}$.

**Problem 3.** Archimedes has a digital weight with a 1 l container placed on top of it. The container weights 250 g. Then, Archimedes filled the container halfway with water, and finally, he threw in a piece of wood with mass 300 g and density $600\,\mathrm{kg/m^3}$, which stayed floating on the water. What mass in grams is displayed on the scale right now?

*Result.* 1050

*Solution.* Even though the wood floats because a buoyant force props it up, this won't affect the measured mass. The scale only measures the total mass of the container and its contents, which is unaffected by forces between the objects within the container (given that these objects are at rest). Therefore, we can simply sum up the masses of the objects: the container weighs 250 g, half a liter of water weighs 500 g, and the piece of wood weighs 300 g. The scale will measure $250\,\mathrm{g} + 500\,\mathrm{g} + 300\,\mathrm{g} = 1050\,\mathrm{g}$.

**Problem 4.** Patric is making his timetable for one day, when he wants to squeeze 3 math hours and 2 physics hours within a 5 hour slot. How many different timetables can he compose?

*Result.* 10

*Solution.* Let us write all possibilities. Start with maths at first place: MMPPP, MPMPP, MPPMP, MPPPM. Now, look at the combinations where the Physics is first: PMMPP, PMPMP, PMPPM, PPMMP, PPMPM, PPPMM. So, there are 10 possible Patric's timetables.

**Problem 5.** Snail is preparing for a long journey. He decided to cross 100-yard long football field. How many hours does this route take if the snail passes to 1 inch distance in 10 seconds?

*Result.* 10

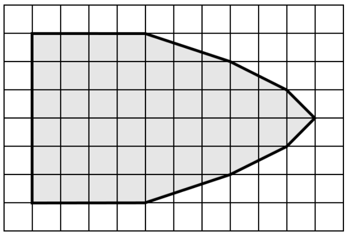
*Solution.* There is 3 feet in 1 yard, and 12 inch in 1 feet. So, 1 yard equals $3 \cdot 12 = 36$ inches. Snail have to pass 100 yards which is equivalent to $100 \cdot 36 = 3\,600$ inches. It takes 10 seconds to pass every inch of field. Therefore, it takes $10 \cdot 3\,600 = 36\,000$ seconds to cross the field. There are 60 minutes in every hour and 60 seconds in every minute. So, 1 hour $= 60 \cdot 60 = 3\,600$ seconds. As a result, snail needs $36\,000 : 3\,600 = 10$ hours to cross the whole field.

**Problem 6.** Joseph multiplied the number 111 111 111 by itself. What is the sum of digits of the resulting number?

*Result.* 81

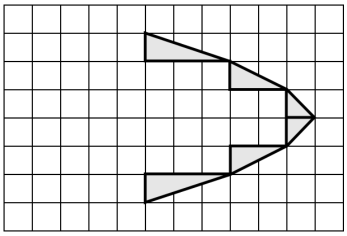
*Solution.* If we multiply the number 111 111 111 by itself we obtain number 12 345 678 987 654 321. Sum of digits of this number is $1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 + 8 + 7 + 6 + 5 + 4 + 3 + 2 + 1 = 81$.

**Problem 7.** Susan drew a projectile on graph paper as you can see in the picture. Filling 1 square on graph paper requires 1 gram of ink. How many grams of ink did she use?



*Result.* 46

*Solution.* Let us first count how many squares are entirely filled. There are 40 entirely filled squares, corresponding to 40 grams of ink. Besides these squares, there are some partially filled squares. When we remove all entirely filled squares from the picture, we can see 6 painted triangles.



All triangles on the same vertical line can be paired to form 3 different rectangles. The first rectangle covers 3 squares; the second covers 2 squares and the third one covers 1 square. Filling them requires $3 + 2 + 1 = 6$ grams of ink. Therefore, Susan needs 46 grams of ink to paint the whole picture.

**Problem 8.** One morning Adam went to an athletic stadium with 400 m long running track. In the first 20 minutes of his run, he ran a very fast to the extent that if he maintained his tempo for the full hour, he would complete the 400 meter track 20 times over. What was Adam's average speed in km/h in the first 20 minutes of his run?

*Result.* 8

*Solution.* If Adam maintained his tempo, he would ran 20 times 400 m and hence $20 \cdot 400\,\mathrm{m} = 8000\,\mathrm{m}$ in one hour. This corresponds to the speed of 8 km/h. What about his average speed? Since Adam did not change his speed during first 20 minutes, 8 km/h must have been his average speed. So, his average speed during first 20 minutes was 8 km/h.

**Problem 9.** City buses operate back-and-forth between two terminal stations such that the passengers never wait for the bus over 10 minutes. What is the minimum number of buses necessary for operating this route, given that it is 7200 m long and the buses drive with an average speed of 5 m/s?

*Result.* 5

*Solution.* A one-way trip between two terminal stations takes

$$\frac{7200\,\mathrm{m}}{5\,\mathrm{m/s}} = 1440\,\mathrm{s} = 24\,\mathrm{min}.$$

Therefore, a return trip of any one bus takes $2 \cdot 24\,\mathrm{min} = 48\,\mathrm{min}$. If we want the passengers to never wait for over 10 min, there must be at least 4 different buses departing the station by the time the first bus arrives. Consequently, five buses are necessary.

**Problem 10.** Jaro has a printer in his office. This printer prints 20 pages per minute. There is, however, another printer in the meeting room capable of printing 25 pages per minute. The printer in the meeting room is connected to the Jaro's computer so that Jaro can send documents to any printer instantly from his office. It takes Jaro 1 minute to walk from his office to the meeting room. What is the least number of pages such that Jaro would be quicker to use the printer in the meeting room instead of the one in his office?

*Result.* 101

*Solution.* When using the printer from the meeting room, Jaro wastes time only returning back to his office, which is one minute. To prefer printing in the meeting room, he needs to print a certain number of pages so that the printer in the meeting room is at least 1 minute faster than the printer from Jaro's office. That means if Jaro starts printing simultaneously the same number of pages in both his office and in the meeting room, at the moment when the printer in the meeting room ends, the printer in the office needs to print for at least one more minute. Notice that the printer from the office prints 20 pages per minute, and the meeting room printer is quicker by $25 - 20 = 5$ pages per minute. Therefore, the printer in the office has to keep printing at least 4 minutes to be 20 pages behind the faster printer, and thus the total number of pages must be more than 100 pages. Therefore, it is quicker for Jaro to use the printer from the meeting room when printing at least 101 pages.

**Problem 11.** Pedro decided to play with his favourite positive integer. First, he rounded it to the tens place, then to hundreds place and finally to thousands. He was very surprised to notice that all three results of rounding were identical but nonzero. What is the smallest possible number that can be Pedro's favourite number?

*Result.* 995

*Solution.* When rounding to tens place, the maximal difference between the result and Pedro's favourite number is 5. The smallest positive integer that can be a result of rounding to thousands is 1 000. So Pedro's favourite number must be at least $1\,000 - 5 = 995$. One can easily notice that 995 satisfies all the requirements.

**Problem 12.** Nina runs around a lake with 6 km long circumference. It took her 40 minutes to run the first lap. After the second lap, she noticed that her average speed during both laps was 8 km/h. How many minutes did it take her to complete the second lap?

*Result.* 50

*Solution.* Nina ran two laps, so the total distance is $2 \cdot 6\,\mathrm{km} = 12\,\mathrm{km}$, with average speed 8 km/h. Therefore, the total time of the run must be

$$\frac{12\,\mathrm{km}}{8\,\mathrm{km/h}} = 1.5\,\mathrm{h} = 90\,\mathrm{min}$$

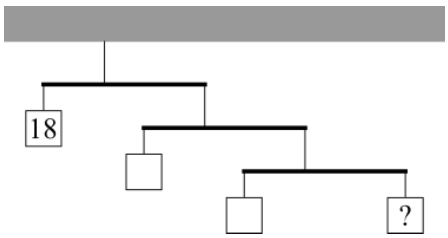
Since Nina ran the first lap in 40 minutes, she had to run the second lap in $90 - 40 = 50$ minutes.

**Problem 13.** Andrew, Beatrice and Constance played chess against each other. No game ended with a tie. Andrew won 7 times and lost 10 times. Beatrice won 8 times and lost 9 times. Constance lost 8 times. How many times did Constance win?

*Result.* 12

*Solution.* The event of someone loosing happened exactly $10 + 9 + 8 = 27$ times – this must be the total number of games played. Therefore, the total number of won games must be 27 as well. This means that Constance had to win $27 - 7 - 8 = 12$ games.

**Problem 14.** Patrick has three lightweight, hangable levers. Every lever's long arm is twice its short arm. Onto the short end of the first lever, Patrick hung an 18 kg weight. Onto the long end of the first lever, he hung the second lever, and onto the second lever's long end he hung the third lever. Then, Patrick hung various weights onto the remaining free ends of the levers, such that the system of levers became balanced. What is the mass in kilograms of the weight which he hung onto the long end of the third lever?




*Result.* 1

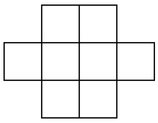
*Solution.* The long arm of every lever is twice its short arm. Therefore, for every lever to be in balance, the mass of everything that is hung on its short end must be twice the mass of everything that is hung on its long end. On the short end of the first lever, there is the 18 kg weight. On the long end, there hangs the second lever, which means that the mass of everything that is hung on the second lever must be $18\,\mathrm{kg} : 2 = 9\,\mathrm{kg}$. On the second lever, we have another weight as well as the third lever. Similarly, for the second lever to be in balance, we need the weight (on the short arm) to be twice as heavy as the third lever (on the long arm). Therefore, that weight will have mass 6 kg and the two weights on the third lever will have a combined mass of 3 kg. Finally, we divide this mass between the two weights in the usual $2 : 1$ ratio. We find that the weight on the long end of the third lever weighs 1 kg.

**Problem 15.** The rain came upon Jane's house. Luckily, Jane has a water collection system on her horizontal roof. The system covers an area of $100\,\mathrm{m}^2$. Throughout the day, the system has collected $4000\,\mathrm{l}$ of water. Jane also has a pool near her house. How many millimeters has the water level risen in Jane's pool after the rain?

*Result.* 40

*Solution.* If the water didn't drain away into the collection system nor overflow the edges of the roof, we would find a block of water on the roof at the rainstorm's end. Assuming the rain to have fallen equally densely over the pool as it did over the roof, the water level increase in the pool will be the same as the height of this imaginary block of water on the roof. The block of water has volume $4000\,\mathrm{l} = 4000\,\mathrm{dm}^3 = 4\,\mathrm{m}^3$. The area of its base is equal to the area of the roof: $100\,\mathrm{m}^2$. We know that the block's volume is the product of its height and its base area. Therefore its height is $\frac{4\,\mathrm{m}^3}{100\,\mathrm{m}^2} = 0.04\,\mathrm{m} = 40\,\mathrm{mm}$. As a result, the water level in the pool has increased by $40\,\mathrm{mm}$.

**Problem 16.** In the attic, Sergey found a strange tablet (see figure). He decided to engrave into each cell one of the numbers from 1 to 8 in such a way that the difference between any two neighboring fields (by edge or corner) is at least 2. When he is done, what is the sum of numbers in cells neighboring (by edge or corner) the cell where the number 4 is engraved?



*Result.* 22

*Solution.* The condition that the numbers in neighboring cells differ by at least 2 means that numbers that differ by only 1 can't neighbor each other.

Let's look at the second and third cell in the middle row. Both cells neighbor all the other cells except for one. What numbers can we place on these cells? Only numbers, that differ by 1 from exactly one other number in the range 1 - 8. Clearly, only numbers 1 and 8 satisfy this condition. Due to the symmetry of the tablet, it doesn't matter in what order we place those numbers, and so we will do it this way:

| | | | |
|---|---|---|---|
| | | | |
| | 1 | 8 | |
| | | | |

Numbers 1 and 8 both have only one other number from which they differ by 1. Those are 2 and 7, respectively. We must therefore place 2 and 7 in the remaining two cells of the middle row.

| | | | |
|---|---|---|---|
| | | | |
| 7 | 1 | 8 | 2 |
| | | | |

The cell with number 2 cannot neighbor the cell with number 3, which therefore has to be above or below number 1. For similar reasons, the number 6 must be located above or below the number 8. This leaves us with two possibilities. If we put both 3 and 6 into one row, in the remaining row, we will have to place numbers 4 and 5 neighboring each other, which is forbidden. Thus we have to go for the second option, e.g., placing 3 and 6 into different rows. Now we notice that 4 and 3 cannot neighbor each other, which leaves exactly one way to fill out the table:

| | | | |
|---|---|---|---|
| | 3 | 5 | |
| 7 | 1 | 8 | 2 |
| | 4 | 6 | |

Finally, we compute that the sum of numbers that neighbor the cell with the number 4 will be 22.

**Problem 17.** Jacob is sitting in the train which is moving $108\,\mathrm{km/h}$. Suddenly the train enters a tunnel, which is, based on Jacob's train book, $2\,\mathrm{km}$ long. Jacob noticed, that the train left the tunnel $75\,\mathrm{s}$ after entering. What is the length of the train in meters?

*Result.* 250

*Solution.* For convenience, let's change the speed from the kilometers per hour to meters per second:

$$108\,\mathrm{km/h} = 30\,\mathrm{m/s}$$

The train with speed $30\,\mathrm{m/s}$ moving for $75\,\mathrm{s}$ passes distance $75\,\mathrm{s} \cdot 30\,\mathrm{m/s} = 2250\,\mathrm{m}$. Therefore, the locomotive passes $2250\,\mathrm{m}$ from entering tunnel. As tunnel is $2000\,\mathrm{m}$ long, after 75 seconds, locomotive was $2250\,\mathrm{m} - 2000\,\mathrm{m} = 250\,\mathrm{m}$ behind the tunnel. The train must be $250\,\mathrm{m}$ long.

**Problem 18.** Ella has six favourite pencils with different lengths whose length is integral in millimeters. The average length of a pencil is 12 mm. What is the greatest possible length (in millimeters) of a pencil that Ella can have?

*Result.* 57 mm

*Solution.* Since the average of Ella's pencils is 12 mm, the sum of their length must be $6 \cdot 12\,\mathrm{mm} = 72\,\mathrm{mm}$. In order to make a given pencil the longest, the other 5 must be as short as possible. Hence they must have lengths 1 mm, 2 mm, 3 mm, 4 mm, 5 mm respectively, which is in total 15 mm. For the longest pencil will remain $72\,\mathrm{mm} - 15\,\mathrm{mm} = 57\,\mathrm{mm}$.

**Problem 19.** Jim is practicing curling on a frozen lake. He takes a curling stone with mass 18.6 kg and sends it sliding on the ice with the initial speed 2 m/s. The frictional coefficient of ice and the stone is 0.05. How far in meters will the stone slide from Jim's hand?

*Result.* 4

*Solution.* Due to the conservation of energy, the stone looses a kinetic energy as a result of an acting force doing a work on the stone. Initially the stone with mass $m = 18.6\,\mathrm{kg}$ and velocity $v = 2\,\mathrm{m/s}$ has a kinetic energy $E_k = \frac{1}{2}mv^2$. The force that slows the stone down is the force of friction $F_t$, which is equal to the product of the normal force from the ice and the frictional coefficient $f$. In this situation, the normal force is the force of Earth's gravity $F_G = mg$ where $g$ is the gravity of Earth. If the stone slides a distance $s$, the force of friction does a work on the stone $W = F_t s = F_G f s = mgfs$. When the stone stops sliding, it has zero velocity and therefore also zero kinetic energy. Then we can calculate the distance $s$ as follows:

$$\begin{aligned}
\Delta E_k &= W \\
\frac{1}{2}m\Delta v^2 &= mgfs \\
\frac{1}{2}(v^2 - 0^2) &= fgs \\
\frac{1}{2}v^2 &= fgs \\
s &= \frac{v^2}{2fg}
\end{aligned}$$

Given the above values, the stone will slide distance of $s = \frac{(2\,\mathrm{m/s})^2}{2\cdot 0.05\cdot 10\,\mathrm{m/s^2}} = 4\,\mathrm{m}$ from Jim's hand.

**Problem 20.** As Kaja was walking along the school corridors, she spotted on one of the board a strange figure. The figure consisted of line segment $BC$ on whose perpendicular bisector there were points $A$ and $D$ such that the point $D$ was inside the triangle $ABC$. The measure of an angle $BAC$ was 40° and the measure of an angle $BDC$ was 140°. What was the measure of an angle $ACD$ in degrees?

*Result.* 50

*Solution.* Since the points $A$ and $D$ are on perpendicular bisector of line segment $BC$, we see that the triangles $ABC$ and $DBC$ are isosceles with base $BC$. Using this fact, we can compute the measures of angles at the base in triangle $ABC$ as $\frac{180^\circ - 40^\circ}{2} = 70^\circ$ and in triangle $DBC$ as $\frac{180^\circ - 140^\circ}{2} = 20^\circ$. Hence the measure of an angle $ACD$ is $|\measuredangle ACD| = |\measuredangle ACB| - |\measuredangle DCB| = 70^\circ - 20^\circ = 50^\circ$.

**Problem 21.** Kate went hiking into the mountains. The weather was cold and so she decided to make a tea. Kate had 0.5 l of water at 0 °C. She brew the tea over a campfire with efficiency of 0.5%. What was the mass of wood in kilograms she needed to use in order to boil the whole volume of water?

*Result.* 2

*Solution.* The heat of combustion of wood is $21\,\mathrm{MJ/kg} = 21\,000\,\mathrm{kJ/kg}$. Since the efficiency of burning wood in a campfire is only 0.5%, by burning one kilogram we get $21\,000\,\mathrm{kJ/kg} \cdot 0.005 \cdot 1\,\mathrm{kg} = 105\,\mathrm{kJ}$ of energy. To get the water to boil, that is, increase its temperature from 0 °C to 100 °C, Kate needs an energy of $(100\,^\circ\mathrm{C} - 0\,^\circ\mathrm{C}) \cdot 4.2\,\mathrm{kJ/(kg\,^\circ C)} \cdot 0.5\,\mathrm{kg} = 210\,\mathrm{kJ}$, which is exactly two times 105 kJ. Therefore, Kate needs 2 kg of wood.

**Problem 22.** Lenka bought a pack of sweets, which contains apple and banana sweets. In a pack, there was twice as many apple sweets as banana sweets. She immediately ate 17 apple and 17 banana sweets. Then, she noticed that there were three times more apple sweets than banana sweets. How many apple sweets was in the pack in the beginning?

*Result.* 68

*Solution.* Denote A as the number of apple sweets and B as the number of banana sweets. In the beginning, there was twice as many apple sweets as banana sweets, which means $A = 2 \cdot B$. After Lenka ate 17 apple and 17 banana sweets, she had three times more apple sweets than banana sweets, which means $A - 17 = 3 \cdot (B - 17)$. Replacing $A$ with $2 \cdot B$ (we know this from the first equation) we get:

$$\begin{aligned} 2 \cdot B - 17 &= 3 \cdot (B - 17) \\ 2 \cdot B - 17 &= 3 \cdot B - 51 \\ B &= 34 \end{aligned}$$

Therefore, there were 34 banana sweets and 68 apple sweets in in the pack initially.

**Problem 23.** Somebody stole the Naboj! It was one of those four people: Majo, Matthew, Jerry or Marcel. Those suspects said the following to the Sheriff:

- Majo: *I don't have the Naboj. Matthew has the Naboj.*
- Matthew: *Marcel has the Naboj. If Marcel has the Naboj, then Majo doesn't have the Naboj.*
- Jerry: *Exactly one out of my sentences is true. Marcel will say either two true, or two untrue sentences.*
- Marcel: *Each suspect said at least one untrue sentence. I do not have the Naboj.*

How many of the sentences which the Sheriff has heard are true? Find the product of all possible answers.

*Result.* 20

*Solution.* First, let's consider Jerry's first sentence. If it was true, it would itself be Jerry's single true sentence, so her second sentence would have to be false. If the first sentence was false, then the second sentence again could not be true - otherwise the first sentence would have to be true as well. So we can say with certainty that Jerry's second sentence is false. But we cannot yet say anything about her first sentence.
Jerry's second sentence is false. Therefore, Marcel must have said one true and one false sentence. Let's discuss what would happen if his first or his second sentence were the true one.
If it were true that "all the suspects have said at least one false sentence", then Marcel's sentence, that he does not have the Naboj, would be false. Therefore, he would have to be the thief. But then, both of Matthew's sentences would have to be true, which would be impossible, since we consider Marcel's sentence "all the suspects have said at least one false sentence" to be correct.
Therefore the true sentence must be Marcel's second sentence, that he does not have the Naboj, and due to his first sentence being false, there must be somebody who has said two true sentences. That can now only be Majo or Matthew. But as Marcel does not have the Naboj, Matthew cannot have said two true sentences. Therefore the one whose both sentences are true is Majo, who says that Matthew has the Naboj. Therefore, Matthew is the thief.
Finally we have to examine how many sentences could be correct. We have proven that both of Majo's sentences were true. Out of Matthew's sentences, only the second one was correct. Similarly, only the second one of Marcel's sentences was correct. We now see that Jerry's first sentence can be either true or false. So there could have been 4 or 5 true sentences. The sum of possible quantities of true sentences is therefore $4 \cdot 5 = 20$.

**Problem 24.** Mary bought herself 8 books, each of the cuboid shape with sides 5 cm, 15 cm and 20 cm. Each of the books had a density of $1200\,\mathrm{kg/m^3}$. The books were packed in a tall box with a lid, placing the books one on another so that the books were touching by the sides with the greatest area. Mary brought the box to her house and placed it on the ground. She opened the lid and now she would like to place the books onto a shelf that is 1.6 m above the ground so that all books are touching the shelf by one of the sides with the smallest area. What work in Joules must Mary do to accomplish that?

*Result.* 216

*Solution.* Books on the floor form a cuboid with a height of $8 \cdot 5\,\mathrm{cm} = 40\,\mathrm{cm}$, thus their center of mass is at the height of $20\,\mathrm{cm} = 0.2\,\mathrm{m}$ above ground. If the books were placed on the shelf, they would form a cuboid with a height of 20 cm and the center of mass at the height $10\,\mathrm{cm} = 0.1\,\mathrm{m}$ above the shelf. Since the shelf is 1.6 m above ground, the difference in the height of the center of mass of the books before and after lifting on the shelf is $1.6\,\mathrm{m} - 0.2\,\mathrm{m} + 0.1\,\mathrm{m} = 1.5\,\mathrm{m}$. Volume of the books is $15\,\mathrm{cm} \cdot 20\,\mathrm{cm} \cdot 40\,\mathrm{cm} = 12\,000\,\mathrm{cm^3} = 12\,\mathrm{dm^3}$ and their density is $1200\,\mathrm{kg/m^3} = 1.2\,\mathrm{kg/dm^3}$. Therefore their mass is $12\,\mathrm{dm^3} \cdot 1.2\,\mathrm{kg/dm^3} = 14.4\,\mathrm{kg}$. The difference in potential energy between having the books on the ground and on the shelf is then equal to $14.4\,\mathrm{kg} \cdot 10\,\mathrm{N/kg} \cdot 1.5\,\mathrm{m} = 216\,\mathrm{J}$. That means Mary has to do a work of 216 J.

**Problem 25.** George carries a triangle, $ABC$, with whole-numbered sides (in centimetres) in his backpack. Side $AB$ is not shorter than 21 cm, but also not longer than 28 cm. Side $AC$ is at least 11 cm, but at most 18 cm long. Side $BC$ is not longer than 8 cm, but it is at least 1 cm long. What is the maximum possible perimeter in centimeters that the triangle could have?

*Result.* 51

*Solution.* They key aspect of this problem is to remember that for any triangle, the triangle inequality holds. In particular, the triangle inequality says that the sum of any two sides of a given triangle must be strictly larger than the third side. Let us examine the the two shorter sides of the triangle: $BC$ and $AC$,. We know that their lengths can be at most 8 cm and 18 cm, respectively. Based on this, we can state that for the triangle to exist at all, its third side must be shorter than $8\,\mathrm{cm} + 18\,\mathrm{cm} = 26\,\mathrm{cm}$, therefore at most 25 cm. The triangle with side lengths of 8 cm, 18 cm and 25 cm fulfils the triangle inequality and so it can exist. What's more, its two shorter sides have their maximum permitted length and its third side has to be shorter than the sum of those two, so the perimeter of $8\,\mathrm{cm} + 18\,\mathrm{cm} + 25\,\mathrm{cm} = 51\,\mathrm{cm}$ is the largest possible.

**Problem 26.** Sabine has bought a new car. The car has an interesting feature: it always transforms the energy of the fuel to the kinetic energy of the car with the same efficiency, no matter, how does the car accelerators. Sabine accelerated in the city from rest to the speed 40 km/h, while she used 700 kJ of energy. Then she went on a motorway, where she accelerated to the speed 120 km/h. How much energy in kilojoules did she use in order to accelerate on a motorway?

*Result.* 5600

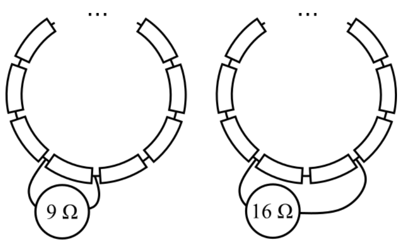
*Solution.* Kinetic energy increases with the second power of the speed. Hence to get 3 times greater speed one has to use $3 \cdot 3 = 9$ times more energy. Thus to accelerate from zero to 120 km/h Sabine needs to use $9 \cdot 700\,\mathrm{kJ} = 6300\,\mathrm{kJ}$ of energy. Therefore, to accelerate from 40 km/h to 120 km/h, Sabine used $6300\,\mathrm{kJ} - 700\,\mathrm{kJ} = 5600\,\mathrm{kJ}$ of energy.

**Problem 27.** Radko wrote all positive integers from 1 to 100 on a piece of paper. Then, on a second piece of paper, he wrote all positive differences of all pairs of numbers from the first piece of paper. Which number appears the most often on the second piece of paper?

*Result.* 1

*Solution.* Let us examine all pairs of numbers from the first paper. We start with pairs that contain number 1. The differences of numbers among these pairs exactly all numbers from 1 to 99. Next, let us look at all pairs that contain 2 but does not contain 1 (or alternatively, pairs where 2 is the smallest number). The differences of numbers among these pairs exactly all numbers from 1 to 98. Similarly we could continue until we look at all pairs where 99 is the smallest number; in such a case, the we only have a single pair whose number difference is 1. In all of these cases, only number 1 was always among the differences, and hence the number 1 appears on the second piece of paper the most often.

**Problem 28.** Nina made a necklace of some same resistors by joining them to make a circle. She connected the multimeter such that there was only one resistor between the clips of the multimeter. The multimeter measured the resistance $9\,\Omega$. When she connected the multimeter such that there were two resistors between the clips of multimeter, the multimeter measured the resistance $16\,\Omega$. How many resistors is Nina's necklace made of?




*Result.* 10

*Solution.* Let's say there are $n$ resistors and each resistor has resistance $R$. When we connect the multimeter on one resistor, we measure resistance in a circuit with paralelly connected resistors. In one branch we have 1 resistor and in the second branch $n-1$ resistors. Similarly if we connect the multimeter on two resistors, we have 2 resistors in one branch and $n-2$ resistors in the second one. This leads us to the system of equations:

$$\frac{1}{9\,\Omega} = \frac{1}{R} + \frac{1}{(n-1)R}$$
$$\frac{1}{16\,\Omega} = \frac{1}{2R} + \frac{1}{(n-2)R}$$

After multiplying the equations by the denominators we get the system:

$$
\begin{aligned}
(n-1)R &= (n-1)(9\,\Omega) + 9\,\Omega \\
(n-2)R &= (n-2)(8\,\Omega) + 16\,\Omega
\end{aligned}
$$

When we expand the parentheses we obtain:

$$
\begin{aligned}
nR - R &= 9n\,\Omega \\
nR - 2R &= 8n\,\Omega
\end{aligned}
$$

Now subtract the second equation from the first one:

$$R = n\,\Omega$$

So the number value of resistance of a resistor in ohms is equal to the number of resistors. Plug this information into the equation $nR - R = 9n\,\Omega$:

$$
\begin{aligned}
(n^2 - n)(\Omega) &= (9n)(\Omega) \\
(n^2 - 10n)(\Omega) &= 0 \\
(n(n-10))(\Omega) &= 0
\end{aligned}
$$

Therefore there are either $n = 0$ or $n = 10$ resistors. The case $n = 0$ however makes no sense, so there must have been $n = 10$ resistors.

**Problem 29.** Laura drew a regular octagon $ABCDEFGH$. She wants to draw 4 non-intersecting line segments (the line segments might not intersect in the endpoints) such that their endpoints will be in the vertices of the octagon. In how many possible ways can she to do that?

*Result.* 14

*Solution.* Distinguish the cases depending on the vertex with which is connected the vertex $A$. If the vertex $A$ is connected with some of the vertices $C$, $E$ or $G$, then this line segment divides the other vertices into two groups with odd number of vertices. However they can't be divided into pairs, so we can't draw line segments satisfying the conditions of the problem. If the vertex $A$ is connected with vertex $B$, we are left with the other 6 vertices to be connected. We have these 5 ways how to do that:

$$(CD)(EF)(GH)$$

$$(CD)(EH)(FG)$$

$$(CF)(DE)(GH)$$

$$(CH)(DE)(FG)$$

$$(CH)(DG)(EF)$$

We would get similar 5 ways, if we connected vertex $A$ with vertex $H$. If we connected the vertex $A$ with vertex $D$, we would immediately have to connect vertices $B$ and $C$. For the other 4 vertices we have these two ways:

$$(EF)(GH)$$

$$(EH)(FG)$$

Analogously we get 2 ways if we connect vertices $A$ and $H$. We have considered all the cases, which vertex can the vertex $A$ be connected with, hence there are exactly $5 + 5 + 2 + 2 = 14$ ways how Laura can draw the line segments.

**Problem 30.** Archimedes took a digital scale and put on it a vessel with volume $1\,\mathrm{l}$ and mass $250\,\mathrm{g}$. Then he filled it with water to a half. Next he put into the vessel with water a pebble on a string such that the whole pebble was under water but it didn't touch the sides of the vessel. The scale showed mass $1\,\mathrm{kg}$. Finally Archimedes put on the scale only the pebble itself. The scale showed again mass $1\,\mathrm{kg}$. What was the density of the pebble in $\mathrm{kg/m^3}$?

*Result.* 4000

*Solution.* If we put a vessel with mass $m_{vessel} = 250\,\mathrm{g}$ on the scale and pour water with volume $V = 0.5\,\mathrm{l}$ in it, the scale shows mass:

$$m_{total} = m_{vessel} + \rho_{water} V$$

After inserting a pebble on string the scale showed mass $m = 1\,\mathrm{kg}$. What caused the increase of mass? Two forces act on the pebble - gravitational and buoyant. It is buoyant force with which the water acts on the pebble. From the law of action-reaction the pebble must act on the water with a force with the same magnitude but with opposite direction. This is the only force, which will change the mass, which the scale shows. The magnitude of force $F$, based on which the scale shows the mass, must then be equal to the sum of magnitude of gravitational force $F_G$ and the magnitude of the buoyancy force of water on pebble $F_b$:

$$\begin{aligned} F &= F_G + F_b \\ mg &= m_{total} g + V_{pebble} \rho_{water} g \\ m &= m_{vessel} + \rho_{water} V + V_{pebble} \rho_{water} \end{aligned}$$

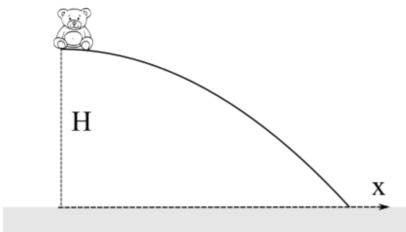
From this we obtain that the volume of pebble is:

$$V_{pebble} = \frac{m - m_{vessel} - \rho_{water} V}{\rho_{water}}$$

From the second weighting we know that the mass of the pebble is $m_{pebble} = 1\,\mathrm{kg}$. Hence the density of the pebble must be:

$$\rho_{pebble} = \frac{m_{pebble}}{V_{pebble}} = \frac{m_{pebble} \rho_{water}}{m - m_{vessel} - \rho_{water} V} = \frac{1\,\mathrm{kg} \cdot 1000\,\mathrm{kg/m^3}}{1\,\mathrm{kg} - 0.25\,\mathrm{kg} - 1000\,\mathrm{kg/m^3} \cdot 0.0005\,\mathrm{m^3}} = 4000\,\mathrm{kg/m^3}$$

**Problem 31.** Alex put his Winnie the Pooh toy on the top of a ramp, whose shape looked like a curve $h(x) = H - ax^2$, such that $a = 0.2/\mathrm{m}$ and $h(0) = H = 2.5\,\mathrm{m}$. However, he has forgotten that ramp is smooth to the extent that the friction is negligible, so Pooh started slide away from him very quickly. What will be the speed of Winnie the Pooh in m/s, when he will be in the horizontal direction $x = 3\,\mathrm{m}$ away from Alex?




*Result.* 6

*Solution.* As Winnie the Pooh slides down, the total mechanical energy preserves. The decrease of potential energy of Pooh must therefore be equal to his kinetic energy. The decrease of the height is $\Delta h = ax^2$. Hence this must hold:

$$\begin{aligned} \Delta E_p &= E_k \\ mg\Delta h &= \frac{1}{2} m v^2 \\ g\Delta h &= \frac{1}{2} v^2 \\ 2g\Delta h &= v^2 \\ v &= \sqrt{2g\Delta h} \\ v &= \sqrt{2gax^2} \\ v &= x\sqrt{2ga} \end{aligned}$$

When we substitute the values from the problem statement, we realise, that Pooh will have speed:

$$v = 3\,\mathrm{m} \cdot \sqrt{2 \cdot 10\,\mathrm{m/s^2} \cdot 0.2/\mathrm{m}} = 6\,\mathrm{m/s}$$

**Problem 32.** Lucas had a white cube. He decided to paint each side either blue, or orange such that no two opposite sides had the same color. Then, he cut the cube into 125 smaller cubes of identical size. How many of those cubes have exactly one blue side and at the same time exactly one orange side?

*Result.* 18

*Solution.* Three out of the original cube's sides had to be painted blue and three had to be orange. What's more, for opposite sides to have opposite colors, all three sides of the same color must have had one common vertex. So let's consider which small cubes can have one blue and one orange side. That kind of cube has to have some edge in common with the original cube - an edge that was shared by one blue and one orange side of the original cube. Along each such edge of the original cube were created 5 smaller cubes. But out of those, 2 also have a common vertex with the initial cube, and so they actually have a third painted side. Therefore, along each such edge of the initial cube where blue and orange sides meet, there will only be $5 - 2 = 3$ small cubes that have exactly 1 blue and exactly 1 orange side. Out of the original cube's edges, there are 6 edges where a blue and an orange side meet, hence there are $6 \cdot 3 = 18$ small cubes of the kind we are looking for.

**Problem 33.** Adam programmed the computer such that when he enters some positive integer, the computer multiplies it 2 021 times by itself and returns number of digits of the result. When Adam entered number 2, the computer returned number 609. When he entered 3, the computer returned 965. Finally when he entered number 4, the computer returned number 1 217. What number will the computer return, when Adam enters number 5?

*Result.* 1 413

*Solution.*
The number, which is the product of $b$ copies of $a$, uses to be donoted as $a^b = \underbrace{a \cdot a \cdot \ldots \cdot a}_{b\text{-times}}$

Notice, that if we multiply 2 021 twos ($2^{2\,021}$) and 2 021 fives ($5^{2\,021}$), we get the same result as if we multiplied 2 021 tens ($10^{2\,021}$). We know how this number looks like - it begins with digit 1 followed by 2 021 zeros, i.e. $1\underbrace{00\ldots0}_{2\,021\text{-times}}$.

The problem states that $2^{2\,021}$ has 609 digits. Hence it is greater than $1\overbrace{00\ldots0}^{608\text{-times}}$ (these two numbers are different, because the number $2^{2\,021}$ is not a multiple of 5, thus it can not end with digit 0). Moreover this number is less than $1\underbrace{00\ldots0}_{609\text{-times}}$. Together we have:

$$1\underbrace{00\ldots0}_{608\text{-times}} < 2^{2\,021} < 1\underbrace{00\ldots0}_{609\text{-times}}$$

Denote by $n$ the number of digits of $5^{2\,021}$. As in the previous paragraph, this number is greater than $1\overbrace{00\ldots0}^{(n-1)\text{-times}}$ (they are different again, because the number $5^{2\,021}$ is not a multiple of 2, so it can not end with digit 0). In addition to this it is less than $1\underbrace{00\ldots0}_{n\text{-times}}$. Therefore it holds that:

$$1\underbrace{00\ldots0}_{(n-1)\text{-times}} < 5^{2\,021} < 1\underbrace{00\ldots0}_{n\text{-times}}$$

Put these things together. Look at the product of $2^{2\,021}$ and $5^{2\,021}$. If we replace the numbers by less ones, we get less product. Hence we have:

$$1\underbrace{00\ldots0}_{608\text{-times}} \cdot 1\underbrace{00\ldots0}_{(n-1)\text{-times}} < 2^{2\,021} \cdot 5^{2\,021}$$

Similarly:

$$2^{2\,021} \cdot 5^{2\,021} < 1\underbrace{00\ldots0}_{609\text{-times}} \cdot 1\underbrace{00\ldots0}_{n\text{-times}}$$

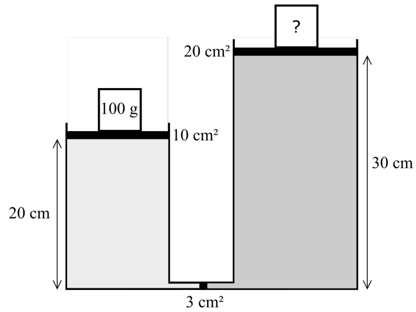
Recall that by multiplication of numbers ending with zeros, the number of zeros is being added, so we get:

$$1\underbrace{00\ldots0}_{(n+607)\text{-times}} < 2^{2\,021} \cdot 5^{2\,021} < 1\underbrace{00\ldots0}_{(n+609)\text{-times}}$$

This way we bounded the number $2^{2\,021} \cdot 5^{2\,021}$ between the number $1\overbrace{00\ldots0}^{(n+607)\text{-times}}$ and the number $1\overbrace{00\ldots0}^{(n+609)\text{-times}}$. The only number within these bounds, which begins with 1 followed by zeroes, is a number with $n + 608$ zeroes. But we know that $2^{2\,021} \cdot 5^{2\,021}$ has this form, hence it must be this number. Since we have already derived the equation $2^{2\,021} \cdot 5^{2\,021} = 1\underbrace{00\ldots0}_{2021\text{-times}}$, it must hold that $n + 608 = 2\,021$.

From this we find the unknown number of digits of $5^{2\,021}$ to be $n = 2\,021 - 608 = 1\,413$.

**Problem 34.** Matej got a hydraulic device which you can see in the figure. It consists of two parts. One is filled with water and the other one is filled with oil. This two parts are connected with movable piston with area $3\,\mathrm{cm}^2$. The part filled with water has another piston with area $10\,\mathrm{cm}^2$ in height $20\,\mathrm{cm}$. The part filled with oil has also another piston but this has area $20\,\mathrm{cm}^2$ and is in height $30\,\mathrm{cm}$. Matej put a weight with mass $100\,\mathrm{g}$ on the piston in the part with water. What must be the mass of a weight in grams Matej needs to put on the piston in the part with oil to make the system to remain at rest?




*Result.* 60

*Solution.* If pistons are not moving, there have to be same force acting on piston from both sides, so the sum of forces is 0. As the piston have the same area at both sides, there must be the same pressure on piston. The pressure acting on small piston from direction of piston with water, have two components- hydrostatic pressure and pressure induced by external force (weight of object on 1. piston). Hydrostatic pressure of the water column with height $h_1 = 20\,\mathrm{cm}$ is:

$$p_{h1} = h_1 \rho_{water} g$$

Pressure induced by object with weight $m_1 = 100\,\mathrm{g}$ putted on piston with area $S_1 = 10\,\mathrm{cm}^2$ is:

$$p_{t1} = \frac{m_1 g}{S_1}$$

Similarly, hydrostatic pressure of oil column with height $h_2 = 30\,\mathrm{cm}$ and pressure induced by object with unknown weight $m_2$ putted on piston with area $S_2 = 20\,\mathrm{cm}^2$ are:

$$p_{h2} = h_2 \rho_{oil} g$$
$$p_{t2} = \frac{m_2 g}{S_2}$$

As we said $p_{h1} + p_{t1} = p_{h2} + p_{t2}$. So we can say:

$$h_1 \rho_{water} g + \frac{m_1 g}{S_1} = h_2 \rho_{oil} g + \frac{m_2 g}{S_2}$$
$$h_1 \rho_{water} + \frac{m_1}{S_1} = h_2 \rho_{oil} + \frac{m_2}{S_2}$$

From here we can express $m_2$:

$$m_2 = S_2 \left( h_1 \rho_{water} - h_2 \rho_{oil} + \frac{m_1}{S_1} \right) = 20\,\mathrm{cm}^2 \cdot \left( 20\,\mathrm{cm} \cdot 1\,\mathrm{g/cm}^3 - 30\,\mathrm{cm} \cdot 0.9\,\mathrm{g/cm}^3 + \frac{100\,\mathrm{g}}{10\,\mathrm{cm}^2} \right) = 60\,\mathrm{g}$$

Matej has to put object with weight $60\,\mathrm{g}$ on second piston.

**Problem 35.** Ten people attended a two-act play in a theater. During the first act, all ten of them were sitting in the front row. However, the group have swapped the seats between the acts. Everyone stayed in the front row, but only two of them were sitting on their original seat. Furthermore, each of the eight people that weren't sitting on their original seat was sitting on a seat of one of their original neighbours. In how many ways they could have swapped seats?

*Result.* 15

*Solution.* Let us first choose the two persons who did not change the seats. These two persons divide the rest of people into three groups - on the left from both fixed seats, in the middle, between the fixed seats and on the right from both fixed seats. In such division, we consider a group of no people a "group" nevertheless. Notice that a member of each group has to sit on a seat that formerly belonged to another member of the same group in order to ensure it was their neighbour's.

Let us now look at one edge of a non-empty group - a seat neighbouring a fixed person's seat or the edge of the row. Since the seat has only one neighbor within the group, the person originally sitting on this seat before the swap took a

place have only one option where to sit. Similarly, the person sitting at the edge of the group after the swap took a place have only one option where (s)he could have been sat before. This, however means that the mentioned couple swapped their seats, creating a new "edge". Therefore in a group, we can pair people who simply have to swap their seats. This means each group must have an even amount of members for us to be able to place them all in accord with the problem statement.
We can subsequently "merge" each such pair into one person, creating 4 "merged persons". The question now remains, in how many ways can we place the fixed persons among them? With the two fixed persons we have 6 positions to place a person to. Therefore we can place the first fixed person in 6 ways and the other fixed person subsequently in 5 ways, giving us $6 \cdot 5 = 30$ options. However so far we've counted each option twice because if we swap the two fixed persons, we won't get a new option. This means the people could have swapped seats in 15 different ways.

**Problem 36.** Jonas has two springs. One with spring constant $3\,\mathrm{N/cm}$ and the other one with spring constant $6\,\mathrm{N/cm}$. He connected them point to point. What is the spring constant of the spring he got in N/cm?

*Result.* 2

*Solution.* When we stretch the connected springs with a force $F$, both springs are stretched with this force. The spring with spring constant $k_1 = 3\,\mathrm{N/cm}$ lengthens by $\frac{F}{k_1}$. Similarly, the spring with spring constant $k_2 = 6\,\mathrm{N/cm}$ lengthens by $\frac{F}{k_2}$. If we denote the unknown spring constant of connected springs as $k$, then the connected springs must lengthen by $\frac{F}{k}$, which must be the sum of elongations of the springs. Hence it must hold that:

$$\frac{F}{k} = \frac{F}{k_1} + \frac{F}{k_2}$$
$$\frac{1}{k} = \frac{1}{k_1} + \frac{1}{k_2}$$
$$k = \frac{k_1 k_2}{k_1 + k_2}$$

So the connected springs have spring constant:

$$k = \frac{k_1 \cdot k_2}{k_1 + k_2} = \frac{3\,\mathrm{N/cm} \cdot 6\,\mathrm{N/cm}}{3\,\mathrm{N/cm} + 6\,\mathrm{N/cm}} = 2\,\mathrm{N/cm}$$

**Problem 37.** Jaro got a board game as a Christmas gift. The game is played on a board, which consists of 2 020 tiles around a circle. He places a token on any tile. Then he plays as follows: in the first turn he moves the token by 2 tiles in clockwise direction, in the second by 4 tiles clockwise, in the third by 6 tiles clockwise and so on, in each turn he moves the token by 2 tiles further clockwise than in the previous move. What is the least number of moves Jaro must perform, until the token again stops on tile where he put it in the beginning?

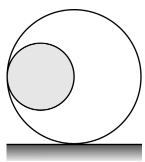
*Result.* 100

*Solution.* After Jaro performs $n$ moves, the token moves by $2 + 4 + 6 + \cdots + 2n$ tiles. When we factor out number two and use an expression for the sum of first $n$ positive integers, we get:

$$2 + 4 + 6 + \cdots + 2n = 2 \cdot (1 + 2 + 3 + \cdots + n) = 2 \cdot \frac{n(n+1)}{2} = n(n+1)$$

In order to get the token after moving by $n(n+1)$ tiles on the tile where it began, the number $n(n+1)$ must be divisible by 2 020. The prime factorisation of the number 2 020 is $2\,020 = 2 \cdot 2 \cdot 5 \cdot 101$. So specially, the number $n(n+1)$ must be divisible by the prime 101. The least $n$ for which it happens is $n = 100$ when $n + 1 = 101$. In that case is also $n = 100$ a factor of $2 \cdot 2 \cdot 5 = 20$. Hence for $n = 100$ the number $n(n+1)$ is divisible by 2 020. Thus we have shown that after 100 moves the token comes to the beginning tile and that this won't happen after less moves. Therefore Jaro performs at least 100 moves.

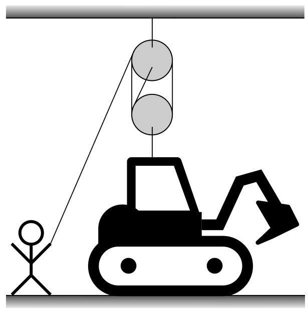
**Problem 38.** Magician Majo magically put a ball with radius 20 cm and mass 0.5 kg into a bigger ball with radius 40 cm and mass 0.5 kg as in the figure. He then dispelled the spell holding the smaller ball in place, and the smaller ball started moving. A while later the smaller ball stopped at the bottom of the bigger ball. How far in centimeters did the bigger ball move from its original contact point with the surface?



*Result.* 10

*Solution.* The only external force acting on each ball is the force of gravity and the normal force from the surface, all acting in vertical direction. Therefore the center of mass of the system cannot move in the horizontal direction. When the system stops moving, the center of mass of the system must be above the contact point of the bigger ball and the surface. We just need to find out how far in horizontal direction from the original contact point is the center of mass of the system located before the movement. The center of mass of the bigger ball is above the contact point with a distance of 0 cm in the horizontal direction. The center of mass of the smaller ball is however in a distance of 20 cm in the horizontal direction. Since the balls have equal mass, the center of the mass of the system is in the midpoint of line segment connecting the centers of mass of the balls. This means the original center of mass of the system is located in a distance of 10 cm horizontally from the original contact point with the surface. The greater ball therefore moves 10 cm from its original contact point with the surface.

**Problem 39.** Bob the builder wants to make work at construction site easier. Dizzy advised him to build a pulley system as in the figure. The key part will be that the rope will go multiple times around the pulleys, so that the rope will not be slipping. Bob can pull the rope with maximal force 800 N and needs to lift Scoop who weights 3500 kg. What is the minimal number of times the rope should go under the movable pulley to lift Scoop?



*Result.* 22

*Solution.* Bob is pulling the rope with a force $F = 800\,\mathrm{N}$. The tension in the rope is therefore equal to the force $F$ in it's every point. This is as well true at the bottom pulley, on both sides of the rope around the bottom pulley. Therefore the rope is acting with a force of $2F$, moving it upwards. This, of course, is true for every time the rope goes around the bottom pulley. If the rope loops around the pulley $n$ times, the acting force on the bottom pulley is $2nF$. To lift Scoop with his mass $m = 3500\,\mathrm{kg}$, this force must be greater than the force of gravity acting on Scoop. Therefore

$$2nF \geq mg$$

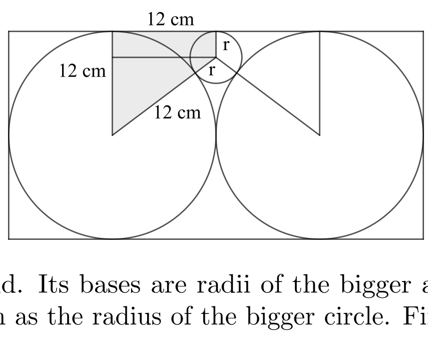
$$n \geq \frac{mg}{2F} = \frac{3500\,\mathrm{kg} \cdot 10\,\mathrm{m/s^2}}{2 \cdot 800\,\mathrm{N}} = 21.875$$

And we can see that the rope musts go around the bottom pulley at least 22 times.

**Problem 40.** Laura wants to draw a new design of Olympic games flag. She drew a rectangle with sides with length 24 cm and 48 cm. Then she drew inside it two circles with radius 12 cm such that these circles were externally tangent. Finally she drew smaller circle, which was tangent to both circles and to the longer side of rectangle. What is the radius of this smaller circle in centimeters?

*Result.* 3

*Solution.* Denote by $r$ the length of radius of the smaller circle. If we draw a figure, we may write down some of the lengths:

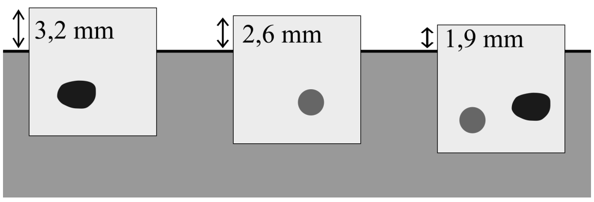



Focus on the highlighted right trapezoid. Its bases are radii of the bigger and the smaller circle. The leg which is perpendicular to the base has also length as the radius of the bigger circle. Finally, the last leg has the length which is sum of the lengths of radii of smaller and bigger circle. Divide this trapezoid on a rectangle and a right triangle. In the right triangle are the lengths of the sides 12 cm, $12\,\mathrm{cm} - r$, $12\,\mathrm{cm} + r$. This triangle is right, so the Pythagorean theorem gives:

$$
\begin{aligned}
(12\,\mathrm{cm})^2 + (12\,\mathrm{cm} - r)^2 &= (12\,\mathrm{cm} + r)^2 \\
144\,\mathrm{cm}^2 + 144\,\mathrm{cm}^2 - r \cdot (24\,\mathrm{cm}) + r^2 &= 144\,\mathrm{cm}^2 + r \cdot (24\,\mathrm{cm}) + r^2 \\
144\,\mathrm{cm}^2 &= 2 \cdot r \cdot (24\,\mathrm{cm}) \\
r &= 3\,\mathrm{cm}
\end{aligned}
$$

This means that the radius of the smaller circle is $r = 3\,\mathrm{cm}$ long.

**Problem 41.** Lucy plays with ice. She takes a small pebble and freezes it into an ice cube. Then she takes a bowl of water and puts the cube on the surface of water. The part of cube above water is 3.2 mm high. Then Lucy takes a small marble and freezes it into an ice cube with the same side length as the previous one. When she puts this cube into water, the part above water is only 2.6 mm high. However Lucy is still not satisfied. Hence she melts both cubes and freezes the pebble and the marble together into a third cube with the same side length as the previous two. When she puts this cube into water, the part above water is 1.9 mm high. What was the side length of all three cubes in millimeters?




*Result.* 39

*Solution.* When Lucy freezes the pebble or the marble into a cube, she doesn't change the volume, however she increases the mass. This changes the average density of the cube. If the cube doesn't sink, the volume of the underwater part of the cube is directly proportional to its average density, therefore also the height of underwater part must be directly proportional to the average density. Moreover, adding of the pebble (or the marble) always increases the average density by the same value. Hence this always increases the height of underwater part of the cube by the same value, which also means that it always decreases the height of the part above water surface by the same value.
When we had in a cube only the pebble and we added the marble, the height of the part above water decreased by $3.2\,\mathrm{mm} - 1.9\,\mathrm{mm} = 1.3\,\mathrm{mm}$. Hence adding of the marble always decreases the height of the part above water by 1.3 mm. If we took the cube with only the marble and took the marble away, the cube containing only ice would have height of the part above water $2.6\,\mathrm{mm} + 1.3\,\mathrm{mm} = 3.9\,\mathrm{mm}$.
For a cube with side length $a$, volume of the underwater part $V'$ and height of part above water $h = 3.9\,\mathrm{mm}$ we moreover get from the Archimedes' principle, that:

$$
\begin{aligned}
mg &= V'\rho_{water} g \\
V\rho_{ice} &= V'\rho_{water} \\
Sa\rho_{ice} &= S(a-h)\rho_{water} \\
a\rho_{ice} &= (a-h)\rho_{water} \\
a(\rho_{water} - \rho_{ice}) &= h\rho_{water} \\
a = h\frac{\rho_{water}}{\rho_{water} - \rho_{ice}} &= 3.9\,\mathrm{mm}\frac{1000\,\mathrm{kg/m^3}}{1000\,\mathrm{kg/m^3} - 900\,\mathrm{kg/m^3}} = 39\,\mathrm{mm}
\end{aligned}
$$

Hence the Lucy's cube had side length 39 mm.

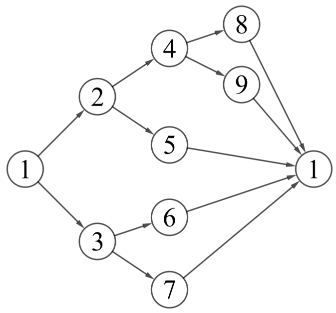
**Problem 42.** Majo started writing a list of numbers: 1, 2, 4, 8, 16, 32 and so on, each number twice the number right above it. This way he has written 555 numbers. Then he created a second list consisting of first digits of numbers from the first list. So the second list was beginning with numbers 1, 2, 4, 8, 1, 3 … and was ending with numbers … 1, 3, 7, 1, 2, 5. Majo noticed that the number 8 is written in the second list 30 times and that the last number in the first list has 167 digits. How many times is the number 9 written in the second list?

*Result.* 24

*Solution.* It looks like there is no pattern in the first digits of numbers in the first list. However, the converse is truth. If we write down some of the first numbers in the second list, we may notice, that the number 1 appears there suspiciously often. From it the numbers increase until they come back to number 1 again.
Let's focus on the information, which numbers can follow any fixed number:

- The number 1 may be followed only by numbers 2 or 3.
- The number 2 may be followed only by numbers 4 or 5.
- The number 3 may be followed only by numbers 6 or 7.
- The number 4 may be followed only by numbers 8 or 9.
- The numbers 5, 6, 7, 8 and 9 may be followed only by number 1 because a ten is carried over.

The possible followers of the numbers are drawn in this figure:




Here we can see that the ones will divide the numbers in the second list into blocks. Moreover, almost all the blocks will be formed by exactly 3 numbers (including number 1). There are only two blocks which contain 4 numbers - block 1, 2, 4, 8 and block 1, 2, 4, 9. In addition, these are the only blocks which contain numbers 8 and 9.
We need to realize just one more thing. When we reach a number 1 in the second list, then the respective number in the first list will have one more digit then the number before it in the first list - it is because the number 1 arise as a first digit after carrying over a ten.
Now we bring everything together. We know that the last three digits in the second list are 1, 2, 5, so we manage to complete the whole block. To the numbers in the first block were corresponding the one-digit numbers in the first list and to the last block were corresponding the 167-digit numbers in the first list. Hence there must be 167 blocks. If all of them consisted of 3 numbers they would have in total $3 \cdot 167 = 501$ numbers. So 4 numbers must be contained in $555 - 501 = 54$ blocks. Thus 54 times must some of the numbers 8 or 9 appear in the second list. Since the number 8 appears 30 times, the number 9 must appear $54 - 30 = 24$ times.